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DISCUSSION

Journal of the
ENGINEERING MECHANICS DIVISION
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PLASTIC ANALYSIS OF SHALLOW CONICAL SHELLS[a]

By E. T. Onat[1]

SYNOPSIS

A complete solution is obtained for a rigid-plastic, simply supported shallow conical shell loaded through a rigid central boss. The solution enables one to assess the effects of changes in geometry on the load carrying capacity of a circular plate. The results are compared with previous theoretical and experimental work.

INTRODUCTION

This paper is concerned with the plastic behavior and the load carrying capacities of shallow conical shells under axially symmetric types of loading and support. The purpose of the paper is two-fold: first a complete explicit solution is obtained for a simply supported conical shell that is loaded through a rigid circular boss, Fig. 1. In the accepted terminology of the theory of rigid, perfectly plastic solids, a complete solution consists in the specification of the critical load intensity at which the shell begins to deform and associated fields of stress and incipient plastic flow. No simplifying assumptions are used in the equations of equilibrium, the yield condition, or the flow rule beyond the starting assumption of a rigid-plastic material that obeys Tresca's yield con-

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[a] The results presented in this paper were obtained in the course of research sponsored by the Office of Ordnance Research, Dept. of the Army, under Contract No. DA-19-020-ORD-4795.

[1] Prof. of Engrg., Brown Univ., Providence, R. I.

dition. Whereas such solutions have been obtained for circular plates and cylindrical shells (1),[2] to the writer's knowledge, no complete solutions were available for shells of more complicated shape. Studies in this field have so far been limited (as of 1960) to obtaining upper and lower bounds to the load carrying capacity of spherical shells (1), (2) and some important types of pressure vessels (3), (4).

The second purpose of the paper is to bring out the importance of the changes of geometry on the load carrying capacity of circular plates. In order to clarify this point, a short digression on the theory of limit analysis will be attempted. A structure composed of rigid, ideally-plastic material and subjected to slowly increasing loads remains rigid until the load intensity reaches a critical value. At this load intensity the structure will start to deform plastically.

It is found that (5) if the subsequent changes in geometry are neglected, then the structure will continue to deform indefinitely while the load intensity remains at this critical value. However, if the changes in geometry are taken into account, then the continuing quasistatic plastic deformation requires either decreasing or increasing loads. The first possibility constitutes a clear case of collapse and the critical load intensity mentioned previously has a substantial physical significance and corresponds to the load carrying capacity of the structure. The second case in general indicates a reserve in strength beyond the critical load intensity: there may be cases where the structure can continue to carry appreciably higher loads without excessive deformation. It should be clear from this discussion that the rational assessment of the influence of changes in geometry on the load carrying capacity is a desirable complement to the theory of limit analysis. It may be noted that the influence of geometry changes has already been considered for rigid-plastic frames (6) and (in an approximate manner) for circular plates (7) and for three dimensional plastic bodies (8), (9).

Experimental results are available for beams (10), arches and plates (11), (12) which show the effects of geometry changes and corroborate the theoretical analysis in some cases (7).

Returning to circular plates we observe that for certain conditions of loading and support the incipient velocity field corresponding to the critical load intensity is such that the plate becomes a shallow circular cone immediately after the start of plastic deformation (13). Thus, the exact solutions offered in this paper may be readily used to assess the effects of geometry changes on the load carrying capacity of plates.

GENERALIZED STRESSES AND STRAINS — EQUATIONS OF EQUILIBRIUM

Fig. 2 shows a shell element of thickness h with the stress resultants transmitted across its boundaries. The function M_ϕ and M_θ are the meridional and circumferential bending moments, N_ϕ and N_θ the meridional and circumferential membrane forces, and Q the shear force. The arrows in Fig. 2 indicate the sign convention employed for forces and couples.

The equilibrium of the shell element free of external loads, requires that

$$\frac{d(sN_\phi)}{ds} - N_\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1a)$$

---

[2] Numbers in parentheses—thus (1)—refer to corresponding items in the Bibliography (see Appendix).

$$N_\theta \tan \phi_o + \frac{d(sQ)}{ds} = 0 \quad \text{(1b)}$$

and

$$\frac{d\left(sM_\phi\right)}{ds} - M_\theta + sQ = 0 \quad \text{(1c)}$$

in which s denotes the distance from the apex of the cone and ϕ_o is the angle of inclination defined in Fig. 2.

For the velocity field of the incipient plastic flow, the usual assumption (Bernoulli-Navier hypothesis) will be made that the particles originally on a normal to the undeformed middle surface continue to remain on a normal to the middle surface as this is deforming.

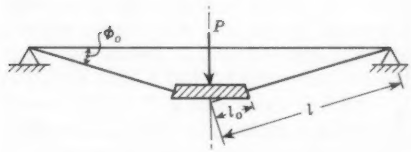

FIG. 1.—SHALLOW CONICAL SHELL WITH A CENTRAL BOSS

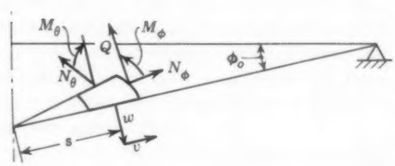

FIG. 2.—SIGN CONVENTION

If the meridional and normal velocity components are denoted by v and w, respectively, the positive directions being those indicated in Fig. 2, the principal rates of extension in the middle surface are

$$e_\phi = \frac{dv}{ds} \quad \text{(2a)}$$

$$e_\theta = \frac{1}{s} \left(v + w \tan \phi_o\right) \quad \text{(2b)}$$

The principal rates of curvature of the middle surface are

$$\chi_\phi = -\frac{d^2w}{ds^2} \quad \text{(3b)}$$

$$\chi_\theta = -\frac{1}{s}\frac{dw}{ds} \quad \text{(3b)}$$

From the point of view of the general theory of limit design (14) the stress resultants M_ϕ, M_θ, N_ϕ, and N_θ are the generalized stresses of the present problem and χ_ϕ, χ_θ, e_ϕ, and e_θ are the corresponding generalized strain rates.

Since the transverse shear Q has little influence on yielding (15) it is not a generalized stress but has the nature of a reaction.

YIELD CONDITION AND FLOW RULE

As with all structural elements, a shell element becomes plastic for certain critical combinations of generalized stresses. These combinations are best expressed in terms of a yield function $f(N_\phi, N_\theta, M_\phi, M_\theta)$ and a positive constant c^2. If the generalized stresses acting on the element are such that, $f < c^2$, then the element remains rigid. On the other hand, if $f = c^2$, that is, the yield condition is satisfied, then the element becomes plastic and may deform plastically. For a shell element composed of ideally plastic material (no strain-hardening) combinations of generalized stresses for which $f < c^2$ are not permissible.

In the case of $f = c^2$, the accompanying plastic deformation (that is, generalized rates of strain) is not arbitrary, but related to the generalized stresses by the equations:

$$e_\phi = \lambda \frac{\partial f}{\partial N_\phi} \quad \text{(4a)}$$

$$e_\theta = \lambda \frac{\partial f}{\partial N_\theta} \quad \text{(4b)}$$

$$\chi_\phi = \lambda \frac{\partial f}{\partial M_\phi} \quad \text{(4c)}$$

and

$$\chi_\theta = \lambda \frac{\partial f}{\partial M_\theta} \quad \text{(4d)}$$

in which λ is an undetermined positive constant. Eqs. 4 are ordinarily referred to as the flow rule.

For the discussion of flow rules and yield conditions, it is convenient to treat the stress resultants M_ϕ, M_θ, N_ϕ, and N_θ as rectangular Cartesian coordinates in a four-dimensional stress space. The generalized strain-rates may be thought of as components of a vector in this space. The yield condition $f = c^2$ represents a closed convex surface (yield surface) in this space. The flow rule (Eqs. 4), in this geometrical interpretation, simply states that the vector with the components e_ϕ, e_θ, χ_ϕ, and χ_θ is in the direction of the outward normal to the yield surface at the stress point considered.

The yield function f is intimately connected with the plastic behavior of the shell material in plane stress. In fact, once this behavior is known and the Bernoulli-Navier hypothesis is adopted for the deformation, then the yield function f can be found by an integration procedure. Such a procedure was developed by E. T. Onat and W. Prager (2) and applied to shells of revolution composed of a rigid-plastic material that obeys Tresca's maximum-shearing stress criterion. Onat and Prager have presented (2) a complete description of the resulting four dimensional yield surface.

For the purposes of the present paper, we shall be interested in the points of this yield surface that give rise to plastic deformations of the following type:

$$e_\phi = 0, \ \chi_\phi = 0, \ e_\theta \neq 0, \ \chi_\theta > 0 \quad \text{(5)}$$

It can easily be shown that such points lie on a definite portion of the yield surface. The mathematical description of this portion involves only the cir-

cumferential stress resultants N_θ and M_θ and has the simple form

$$m_\theta = 1 - n_\theta{}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

in which the bending moments and membrane forces are made dimensionless by dividing them by $M_o = \sigma_o h^2/4$ and $N_o = \sigma_o h$, respectively, and the letters m and n are used, with appropriate subscripts, to denote the resulting dimensionless quantities. Here M_o and N_o are the maximum plastic moment and membrane force, respectively, which a cross section can carry and σ_o is the yield stress of the shell material in pure tension.

As remarked previously, the meridional stress resultants n_ϕ and m_ϕ do not occur in Eqs. 5. However, they are not entirely arbitrary and must satisfy the following inequalities

$$2\left[\left(n_\phi - \frac{n_\theta}{2}\right)^2 - \frac{n_\theta^2}{4}\right] \leq m_\phi \leq 1 - 2\left[\frac{n_\theta^2}{4} + \left(n_\phi - \frac{n_\theta}{2}\right)^2\right] \quad \ldots\ldots \quad (7)$$

It may be remarked that Eqs. 6 and 7 may be obtained independent of the previously mentioned work (2) by considering the strain-rate distribution across the thickness of the shell (which follows from Eqs. 5 and the Bernoulli-Navier hypothesis) and using Tresca's yield condition and the associated flow rule. However, if it is desired simply to use Orat and Prager's (2) tables for deriving Eqs. 6 and 7, then we may indicate that Eq. 6 is obtained from the first line of Table 2 of that work and Eq. 7 from the first two lines of their Table 1 by replacing the subscripts 1 and 2 by θ and ϕ respectively.

For future convenience, inequalities of Eq. 6 may be written in the following form

$$-\frac{1}{2} + 2n'^2 \leq m' \leq \frac{1}{2} - 2n'^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

in which

$$m' = m_\phi + \frac{n_\theta^2}{2} - \frac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9a)$$

and

$$n' = n_\phi - \frac{n_\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9b)$$

In a (m', n') plane, the above inequalities represent a domain bounded by two symmetric parabolic arcs, Fig. 3.

A COMPLETE SOLUTION

Consider the conical shell shown in Fig. 1. The load P acts on a rigid circular boss to which the shell is built in. At the supports, frictionless sliding action is allowed. The complete solution of the problem consists in the specification of the critical load intensity, P^*, and associated fields of stress and incipient plastic flow. These fields are specified by giving the stress resultants M_ϕ, M_θ, N_ϕ, and N_θ, and the velocity components v and w, as functions of the arc length s along the generator.

The stress resultants must satisfy the equations of equilibrium, Eqs. 2. Moreover, the state of stress at a generic point of the shell must be presented by a point on, or inside the yield surface ($f \leq c^2$). In the first case the strain rate components must satisfy the flow rule, in the second case they must van-

ish. The field quantities M_ϕ, M_θ, N_ϕ, N_θ and v and w, and their derivatives with respect to s may exhibit various types of discontinuities. A systematic discussion of these discontinuities will not be attempted here. It is sufficient for the present purpose to mention only that w must be continuous and have piecewise continuous first and second derivatives, and v, together with its first derivative, must be piecewise continuous. The continuity of w follows from the assumed lack of shear action. The discontinuities in v, dw/ds and d^2w/ds^2 are not arbitrary but related to the existing state of stress through the flow rule as will be seen from the following discussion.

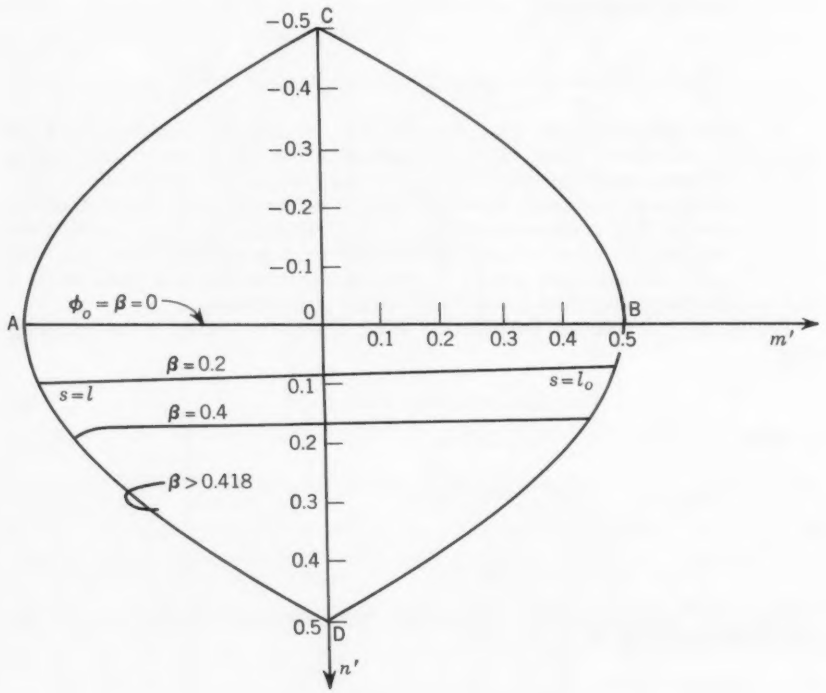

FIG. 3.—STRESS PROFILES IN (m^1, n^1) PLANE

The boundary conditions which must be satisfied by the field quantities are that $N_\phi = 0$, $M_\phi = 0$, $w = 0$ for $s = l$ and

$$P^* = 2\pi \left[Q \cos \phi_o + N_\phi \sin \phi_o \right] l_o \cos \phi_o \text{ for } s = l_o \quad \ldots\ldots \quad (10)$$

Eq. 10 follows from equilibrium. Further conditions that must be satisfied at the built-in edge of the shell will be mentioned later.

The motivation for the construction of the present complete solution may be derived from a study of the special case $\phi_o = 0$. In this case the cone degen-

erates into circular annulus and the following complete solution is easily obtained (13):

$$n_\phi = n_\theta = 0; \quad m_\theta = 1, \quad m_\phi = \frac{1}{1-\alpha}\left(\frac{R_o}{s} - \alpha\right) \qquad (11)$$

$$v = 0 \qquad (12a)$$

$$w = \frac{w_o}{1-\alpha}\left(1 - \frac{s}{R}\right) \qquad (12b)$$

$$P_p^* = 2\pi \frac{M_o}{1-\alpha} \qquad (12c)$$

in which $\alpha = R_o/R$ and R_o and R stand for the interior and exterior radii of the plate respectively. In Eq. 11 w_o is an undetermined positive constant.

It is seen from Eqs. 6 and 7 that the states of stress defined by Eq. 11 fall on the previously discussed portion of the four-dimensional yield surface.

In fact, the mapping of the stress states of Eq. 11 onto the (m', n') plane results in the straight line A B in Fig. 3, in which A and B correspond to states of stress at $s = R$ and $s = R_o$ respectively.

Now it is natural to expect that for shallow cones such as the one shown in Fig. 1, the stress points will mostly lie on the same portion of the yield surface. It may be remarked that the stress points corresponding to $s = l_o$ and $s = l$ are already on the boundary of the domain of interest so that they may fall out of this domain for non-zero ϕ_o. However, it will be seen that this does not happen if $\phi_o < g(\alpha)$, in which g is to be determined later.

Thus, the states of stress and rates of strain to be considered here satisfy Eqs. 6 and 7 and the associated flow rule

$$e_\phi = 0, \quad \chi_\phi = 0, \quad \frac{N_o\, e_\theta}{M_o\, \chi_\theta} = 2n_\theta, \quad \chi_\theta > 0 \qquad (13)$$

From the first two of Eqs. 13 and from Eqs. 2 and 3 we find that

$$v = v_o \qquad (14a)$$

and

$$w = w_o \frac{l-s}{l-l_o} \qquad (14b)$$

in which v_o and w_o are constants and the boundary condition $w = 0$ at $s = l$ is already incorporated in Eqs. 14.

If the axial velocity of the boss is denoted by w_b, then the continuity of w requires that at $s = l_o$

$$w_b \cos \phi_o = w_o \qquad (15)$$

It is immediately noticed that there is a discontinuity in v at $s = l_o$ by an amount $[v]$:

$$[v] = v_o + w_b \sin \phi_o \qquad (16a)$$

or, using Eq. 15,

$$[v] = v_o + w_o \tan \phi_o \qquad (16b)$$

Similarly we observe that dw/ds is discontinuous at $s = l_o$, the discontinuity in - dw/ds amounting to

$$\left[- \frac{dw}{ds}\right] = \frac{w_o}{1-l_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

In view of Eqs. 2 and 3 the existence of these discontinuities indicates that both e_ϕ and χ_ϕ are infinite at the built-in edge of the shell, the ratio of these infinite quantities being equal to the ratio of discontinuities in v and - dw/ds:

$$\left(\frac{e_\phi}{\chi_\phi}\right)_h = \left[\frac{v_o}{w_o} + \tan\phi_o\right] \left(1 - l_o\right) \text{ at } s = l_o \quad \ldots\ldots\ldots\ldots \quad (18)$$

Under these circumstances, the built-in edge of the shell is said to behave as a hinge circle. The subscript h is used to indicate this fact.

Now the question arises whether such a hinge circle is compatible with the stress points falling on the portion of the yield surface singled out in this investigation. Clearly for all states satisfying Eqs. 6 and 7 (Eq. 7 without equality signs) $e_\phi = 0$, $\chi_\phi = 0$ and thus these states cannot give rise to a hinge. However the states of stress

$$n_\phi = n_\theta;\; m_\phi = m_\theta;\; m_\theta = 1 - n_\theta^2 \quad \ldots\ldots\ldots\ldots \quad (19)$$

which falls on the boundary of the portion of interest and corresponds to a point on the arc CBD in (m', n') plane, obviously permits the strain-rates of the type for which

$$\frac{N_o\, e_\theta}{M_o\, \chi_\theta} = \frac{N_o\, e_\phi}{M_o\, \chi_\phi} = 2\, n_\phi = 2\, n_\theta;\; \chi_\phi > 0,\; \chi_\theta > 0 \quad \ldots\ldots \quad (20)$$

Note that this state of stress allows χ_ϕ and e_ϕ to be infinite. On the other hand we find from Eqs. 13 and 14 that

$$\frac{N_o\, e_\theta}{M_o\, \chi_\theta} = \frac{N_o}{M_o}\left[\frac{v_o}{w_o}\left(1 - l_o\right) + \tan\phi_o\,(1 - s)\right] \text{ for } l_o \leq s \leq 1 \quad \ldots\ldots \quad (21)$$

Comparing Eqs. 18, 20 and 21 we conclude that the state of stress previously given (Eq. 18) is a proper one for describing the plastic hinge. Note that the first two equations of Eq. 19 constitute two additional boundary conditions for the stresses.

Now we consider the states of stress for $s > l_o$ and observe from Eq. 21 and the flow rule that

$$2\, n_\theta = \frac{N_o}{M_o}\,\frac{e_\theta}{\chi_\theta} = \frac{N_o}{M_o}\left[\frac{v_o}{w_o}\left(1 - l_o\right) + (1 - s)\tan\phi_o\right] \quad \ldots\ldots \quad (22)$$

Substituting Eq. 22 for n_θ into the first equilibrium equation and using the boundary conditions of Eqs. 9 and 19, we obtain, by simple integration

$$\frac{v_o}{w_o} = -\frac{1+l_o}{2l}\tan\phi_o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

and

$$n_\phi = \beta\left[1 + \alpha^2 - \frac{\alpha\, l_o}{s} - \frac{s}{l}\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (24a)$$

$$n_\theta = \beta \left[1 + \alpha^2 - 2\frac{s}{l}\right] \qquad (24b)$$

in which

$$\beta = \frac{1}{4}\frac{N_o}{M_o} l \tan\phi_o = \frac{l}{h}\tan\phi_o \qquad (25a)$$

and

$$\alpha = \frac{l_o}{l} \qquad (25b)$$

Note that the flow rule demands $\chi_\theta > 0$ and hence $w_o > 0$. Eq. 23 then requires that $v_o < 0$ as would be expected. Now using Eq. 24a and integrating the remaining equations of equilibrium, with the boundary conditions of Eqs. 9, 10 and 19, we complete the solution:

$$m_\theta = 1 - \beta^2\left[1 + \alpha^2 - 2\frac{s}{l}\right]^2 \qquad (26a)$$

$$m_\phi = \alpha\frac{l}{s}\left[\frac{1}{1-\alpha} + \frac{\beta^2}{3}(1 + \alpha + 3\alpha^2 + 3\alpha^3)\right]$$

$$- \frac{\alpha}{1-\alpha} - \frac{\beta^2}{3}(4 + \alpha + 13\alpha^2 + 4\alpha^3 + 3\alpha^4)$$

$$+ \frac{\beta^2}{3}\left[-8\frac{s^2}{l^2} + 12(1+\alpha^2)\frac{s}{l}\right] \qquad (26b)$$

and

$$P^* = 2\pi M_o\left[\frac{1}{1-\alpha} + \frac{\beta^2}{3}(1 + \alpha - 5\alpha^2 + 3\alpha^3)\right]\cos^2\phi_o \qquad (27a)$$

or

$$P^* = 2\pi M_o\left[\frac{1}{1-\alpha} + \frac{\beta^2}{3}(1 + \alpha - 5\alpha^2 + 3\alpha^3)\right]\frac{1}{1 + \beta^2\dfrac{h^2}{l^2}} \qquad (27b)$$

Having thus obtained the solution, we may now check whether the mapping of the stress states given by Eqs. 24 and 26 falls on the domain bounded by two parabolic arcs in Fig. 3.

Such mappings are shown in Fig. 3 for shells with fixed $\alpha = l_o/l = 0.1$, but varying cone angles or β's. It is seen that for β - 0.2 and 0.4 stress points fall within the desired area. It is easily established that for $\beta > 0.418$, the points corresponding to the neighborhood of s = l leave the domain of interest as indicated in Fig. 3. The limiting value of β may be found by determining the tangents to the parabolic arc and to the mapping curve at the point corresponding to s = l and comparing them in an obvious manner.

It is found that for

$$\beta^2 \le \frac{3}{2}\frac{\alpha}{1-\alpha}\;\frac{1}{1 - \frac{\alpha}{2}(1+\alpha) + \frac{3}{2}(\alpha^4 - \alpha^3)} = D(\alpha) \qquad (28)$$

the solution given by Eqs. 14, 24, 26, and 27 is a valid one. For $\beta^2 > D(\alpha)$ the present solution is no longer valid. It is to be noted that for $\alpha = 0$, or for a vanishingly small boss, the solution presented is valid only for $\beta = 0$ or for the plate. It should also be noted that for values of α near unity Eq. 28 does not

constitute the decisive criterion for the validity of the previously described complete solution. The discussion of the case $\alpha \approx 1$ will not be given here.

POST-YIELD BEHAVIOR OF CIRCULAR PLATES

The complete solution described in the preceding section will now be used to discuss the post-yield behavior of simply supported rigid-plastic plates loaded through a circular rigid boss in the center. It is seen from Eq. 12 that such a plate begins to deform plastically when the load reaches the critical value $P = P_p^* = 2\pi \frac{M_o}{1-\alpha}$. The incipient velocity field is given by Eq. 11. It is seen from this equation that the plate becomes a shallow cone through the action of the incipient plastic flow. This cone has an infinitesimally small angle of inclination ϕ_o, or β. The parameter $\alpha = l_o/l$ retains its original value R_o/R. In order to deform the plate (which now is a shallow cone) further, the magnitude of the load must be brought to the critical value given by Eq. 27b. Since β is small and $h/l \cong h/R$, Eq. 27b can be written in the form

$$P = P^* = 2\pi M_o \left[\frac{1}{1-\alpha} + \frac{\beta^2}{3}\left(1 + \alpha - 5\alpha^2 + 3\alpha^3 - \frac{3h^2}{R^2(1-\alpha)}\right)\right] \quad \ldots\ldots (29)$$

We first note that for structures that can be called plates $\frac{h}{R(1-\alpha)} \ll 1$. Moreover, we are interested in relatively small boss sizes so that $\alpha < 1$. Under these limitations, it is seen from Eq. 29 that $P^* > P_p^*$. Thus, the continuing plastic deformations of the plate require increasing loads.

The remarkable aspect of the complete solution of the preceding section is that the incipient velocity field Eq. 14 and Eq. 23 maintains the conical shape. Hence, if we assume that a small overhang at the supports hinders the cone leaving the support (since $v_o < 0$, without overhang the cone should lose contact with the support) and that the influence of the small overhang on the present solution is negligible, then we can follow the continuing deformations of the plate until the present solution breaks down or equivalently the critical value of $\beta \left[\beta^2 = D(\alpha)\right]$ is reached.

For plates with $\alpha \ll 1$ we have from Eq. 29

$$\frac{P}{P_p^*} = 1 + \frac{\beta^2}{3} \left(\text{for } \beta \leq D(\alpha)\right) \quad \ldots\ldots\ldots\ldots\ldots (30)$$

in which the terms in α^2 and $h^2/R^2(1-\alpha)$ are neglected in comparison with unity.

Since $\beta = \tan \phi_o \frac{1}{h} = \frac{\delta}{h}$, in which δ stands for the vertical motion of the boss, β in Eq. 30 may be replaced by $\frac{\delta}{h}$.

It is interesting to compare this result with a previous approximate analysis (7). There it was found that

$$\frac{P}{P_p^*} = 1 + \frac{4}{3}\left(\frac{\delta}{h}\right)^2 \left(\text{for } \frac{\delta}{h} < \frac{1}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots (31)$$

It is seen that the two expressions are quite similar in form, but Eq. 31 overestimates the load considerably. Finally it may be interesting to compare Eq. 30 with experimental results in (12), where among other types of tests, simply supported mild steel plates were loaded with a central rigid punch, the ratio of

the punch diameter to plate diameter being 0.1. Since the plate material right under the punch did not show appreciable plastic deformation, the punch may be thought of as equivalent to a boss with α = 0.1. Fig. 4 shows load-deformation curves for plates with diameter-thickness ratios of 10, 20 and 40. The heavy solid line indicates the present rigid-plastic solution. The dotted extension of the parabola of Eq. 30 is added for further comparison. It may be seen that a superposition of elastic deformations on the curve corresponding to the present

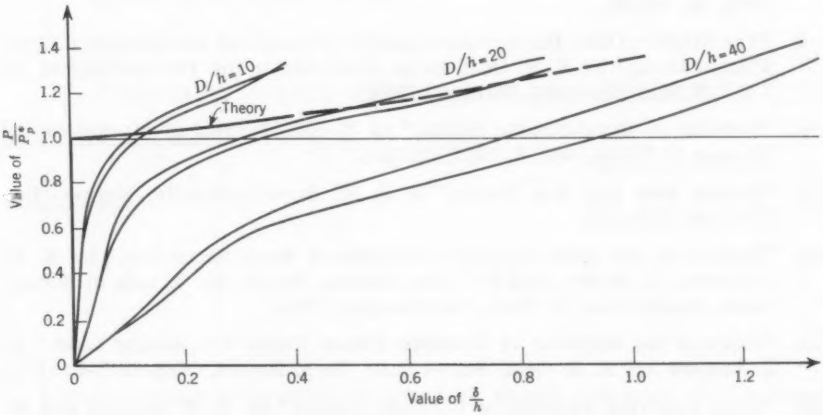

FIG. 4.—COMPARISON WITH EXPERIMENTS

solution provides rather acceptable approximations to actual load-deformation curves for D/h = 20 and 40. For D/h = 10 the actual curves are substantially above the curve provided by the present solution.

APPENDIX.—BIBLIOGRAPHY FOR PLASTIC ANALYSIS OF SHELLS

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END-FIXITY EFFECT ON VIBRATION AND INSTABILITY

By David Burgreen[1]

SYNOPSIS

A study is made of the effect of arbitrary elastic end-fixity on the instability of columns and on the frequency of vibration of beams. The significance and influence of negative end-fixity on buckling and vibration is examined. The analysis is then extended to vibrating columns. Some similarities and special features of these eigenvalue problems are discussed and simple expressions relating instability load and frequency of vibration to end-fixity are proposed.

INTRODUCTION

A number of analyses have been made of the effect of end-fixity on the vibration of beams, the effect of end-fixity on the instability of columns, and the relationship between natural frequency of vibration and buckling for columns with several specified end-fixities.[2,3] This work is extended here to vibration of beams and instability of columns under the influence of negative end-fixities and to the vibration of columns with elastic rotational restraint.

Note.—Discussion open until May 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. EM 6, December, 1960.

[1] Cons. Engr., Nuclear Development Corp. of America, White Plains, N. Y.

[2] "Les Relations entre les Modes Normaux de Vibration et la Stabilité des Systèmes Elastiques," by C. Massonnet, Université de Liège, Bulletin des Cours et des Laboratories d'Essais des Construction du Génie Civil et d'Hydraulique Fluiviale, Vol. 1, 1940.

[3] "Lateral Vibrations as Related to Structural Stability," by H. Lurie, Journal of Applied Mechanics, Vol. 19, 1952.

It is known that for a pin-ended vibrating column, an exact linear relationship exists between the axial load on the column and the square of the natural frequency of vibration. It has also been shown by C. Massonnet[3] that a near-linear relationship exists between the end load and the frequency of vibration when the end restraints are; pinned-pinned, clamped-clamped, pinned-clamped, free-clamped, pinned-guided, and clamped-guided. The question of the validity of the near-linear relationship for any arbitrary elastic rotational restraint, as well as the magnitude of the error involved in the linear approximation, is examined here.

Before treating this more complex eigenvalue problem, we will examine some of the simpler physical systems that are contained in the vibrating column system. We first examine the instability of a weightless beam without an end load. It is found that such a beam does become unstable when the end restraint attains certain negative values. End compressive loads are then added, and the instability problem is treated in relation to the end restraints. The axial loads are subsequently removed, and the vibration problem (now with distributed weight) is investigated, again from the point of view of how changing the end-fixities affects the vibration. Finally we examine the behavior of the vibrating column whose limit solutions are the solutions of the previous problems.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, in the Appendix.

WEIGHTLESS BEAM

The equation of the vibrating column with the end load removed and the mass reduced to zero is

$$\frac{d^4 y}{dx^4} = 0 \quad \text{. . . . .} \quad (1)$$

in which x is the axial distance measured from the center of the member and y is the lateral displacement of beam or column. The boundary conditions for the element, shown in Fig. 1 and signifying arbitrary end-fixity, are

$$y = 0 \qquad \text{at } x = \pm \frac{L}{2}$$

in which L is the length of the beam or column, and

$$\frac{d^2 y}{dx^2} = \alpha \frac{dy}{dx} \quad \left(\text{at } x = -\frac{L}{2}\right) \quad \text{. . . . .} \quad (2a)$$

$$\frac{d^2 y}{dx^2} = -\gamma \frac{dy}{dx} \quad \left(\text{at } x = \frac{L}{2}\right) \quad \text{. . . . .} \quad (2b)$$

in which α and γ are the end fixities at the left and right ends of the member, respectively.

When these boundary conditions are set into the mode form, derived from Eq. 1, the characteristic equation

$$(\alpha L)(\gamma L) + 4(\alpha L + \gamma L) + 12 = 0 \quad \text{. . . . . . .} \quad (3)$$

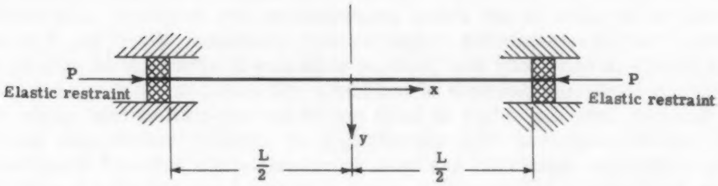

FIG. 1.—CONFIGURATION OF VIBRATING COLUMN WITH ENDS HAVING ELASTIC ROTATIONAL RESTRAINT

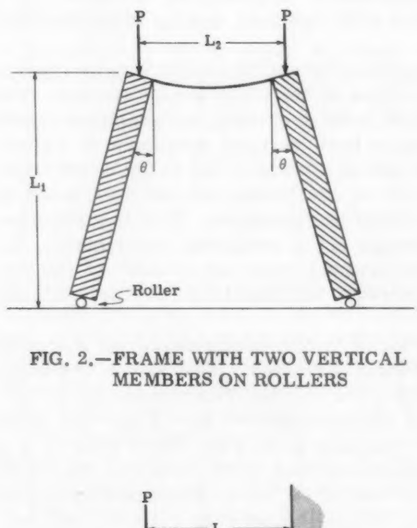

FIG. 2.—FRAME WITH TWO VERTICAL MEMBERS ON ROLLERS

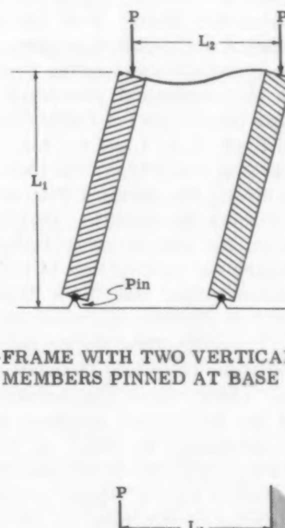

FIG. 3.—FRAME WITH TWO VERTICAL MEMBERS PINNED AT BASE

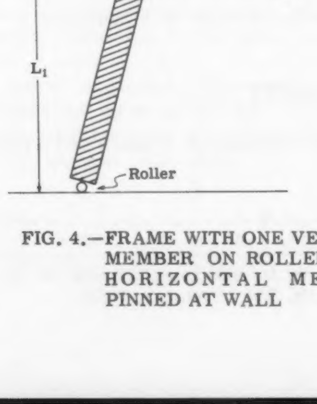

FIG. 4.—FRAME WITH ONE VERTICAL MEMBER ON ROLLERS AND HORIZONTAL MEMBER PINNED AT WALL

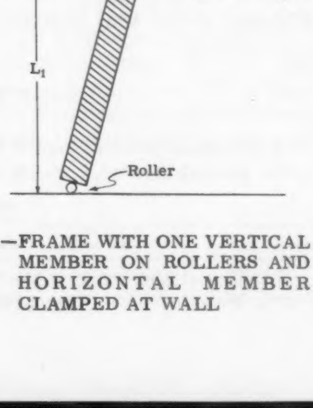

FIG. 5.—FRAME WITH ONE VERTICAL MEMBER ON ROLLERS AND HORIZONTAL MEMBER CLAMPED AT WALL

is obtained. Eq. 3 represents a set of conjugate hyperbolas in the nondimensional variables α L and γ L, and will be seen to be the characteristic equation that is common to the fields generated by the frequency characteristic equations and the instability characteristic equations. Since Eq. 3 gives the combinations of end-fixity that produce solutions of arbitrary amplitude, it indicates that these combinations of end-fixity will result in buckling of the beam. It is apparent from Eq. 3 that at least one of the end-fixities that cause instability will be negative. The significance of negative end-fixities and their ability to produce instability has been discussed by the author.[4] When the fixities at both ends of the beam are equal, it is found that buckling occurs in the symmetrical mode at $\alpha L = \gamma L -2$, and that buckling occurs in the antisymmetrical mode at $\alpha L = \gamma L = -6$. If one end is pinned, buckling will occur when the other end has an end-fixity of -3, and if one end is clamped, buckling takes place when the other end attains an end-fixity of -4. Eq. 3, which gives the foregoing results, is shown as a limiting curve in Figs. 6 and 7, designated as $\omega/\omega_e = 0$ and $P/P_e = 0$, in which ω is the circular frequency of vibration of the beam or column, ω_e is the circular fundamental frequency of vibration of a pin-ended beam, P is the compressive axial end load, and P_e is the buckling load of a pin-ended column.

Negative end-fixities capable of producing instability are found to be present in the horizontal members of certain types of vertically loaded frames. The four conditions for instability enumerated in the preceding paragraph are shown in Figs. 2, 3, 4 and 5. All of these frames have vertical members of infinite bending rigidity. Note that when one end of a vertical leg is displaced horizontally, the moment that acts at the end of the horizontal member tends to increase the rotation that the displacement has produced. This is a requirement for end-moment instability, although not a sufficient requirement. In beams not susceptible to instability, the moment would act counter to the rotation rather than in a direction to increase it. This is, of course, the case with all positive end-fixities.

Consider first the frame in Fig. 2, which is the configuration of a typical movable hoist whose vertical legs are rigid A frames. It buckles in the manner shown when the negative end-fixity is $-\alpha = 2/L_2$ The moment at the end of the horizontal member is $PL_1\theta$ and the curvature $-PL_1\theta/EI_2$. The ratio of curvature to slope is simply $-PL_1/EI_2$ and when $PL_1/EI_2 = 2/L_2$ or $P = 2\,EI_2/L_1L_2$ the frame will buckle. In a similar way, it is found that the frame shown in Fig. 3, which is constrained to buckle in the antisymmetrical mode because of the pins at the base of the frame (rather than rollers), will become unstable when $P = 6\,EI_2/L_1L_2$. The frame in Fig. 4 with the horizontal member pinned at the wall and the vertical member on rollers will buckle when $P = 3\,EI_2/L_1L_2$, and the frame in Fig. 5 with the horizontal member clamped at the wall and the vertical member on rollers will buckle at $P = 4\,EI_2/L_1L_2$.

END-FIXITY AND INSTABILITY

A compressive end load P is now added to the element. A balance of forces in the lateral direction yields the equation

$$\frac{d^4y}{dx^4} + \frac{P}{EI}\frac{d^2y}{dx^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (4)$$

[4] "Effect of End-Fixity on the Vibration of Rods," by D. Burgreen, Journal of the Engrg. Mechanics Div., Proceedings, ASCE, Vol. 84, No. EM 4, October, 1958.

Using boundary conditions 2, we find the characteristic instability equation relating the end load and the end-fixities to be

$$4\left(\frac{pL}{2}\right)^2 - \left(\frac{pL}{2}\right)(\alpha L + \gamma L)\left(\cot\frac{pL}{2} - \tan\frac{pL}{2} - \frac{1}{\frac{pL}{2}}\right)$$

$$-(\alpha L)(\gamma L)\left(1 - \frac{\tan\frac{pL}{2}}{\frac{pL}{2}}\right) = 0 \quad \ldots\ldots\ldots \quad (5)$$

in which

$$p = (P/EI)^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6a)$$

or

$$\frac{P}{P_e} = \left(\frac{pL}{\pi}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6b)$$

If the fixity is the same at either end of the beam, the instability criteria for buckling in the symmetrical and antisymmetrical modes are, respectively,

$$\alpha L = \gamma L = -pL\cot(pL/2) \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$\alpha L = \gamma L = \frac{pL}{\left[\cot\left(\frac{pL}{2}\right) - \frac{1}{\left(\frac{pL}{2}\right)}\right]} \quad \ldots\ldots\ldots \quad (8)$$

It has been pointed out[5] that the characteristic solution for buckling of a clamped-clamped beam in the antisymmetrical mode is often omitted as a result of starting with a second order moment balance equation rather than a fourth order lateral force balance equation. The antisymmetrical buckling load for a clamped-clamped column is included in this solution and may be obtained from Eq. 8 by letting αL go to infinity.

For a constant value of end compressive load, Eq. 5 will generate hyperbolas in the variables αL and γL. The limiting hyperbola in this set is obtained when the end load is reduced to zero yielding the characteristic curve for the weightless beam. Fig. 6 shows the combination of end-fixities and end load that will produce instability. When the fixity at one end of the column is less than -4, buckling will take place regardless of the magnitude of the end compressive load. It is apparent, however, that end-fixities less than -4 may not produce buckling if the load on the column is tensile rather than compressive. Buckling in the symmetrical mode at $P/P_e = 4$ is represented in Fig. 6 by a point at $\alpha L = \gamma L = \infty$, or the clamped-clamped condition. If a pin support were placed at the center of the column to induce buckling in the antisymmetrical mode at $P = 4P_e$, it is found that a linear relationship exists between end-fixities that will produce buckling. It is simply that the sum of the end-fixities add up to zero.

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[5] "A Note on the Buckling of Struts," by H. Lurie, Journal of the Royal Aeronautical Soc., Vol. 55, March, 1951.

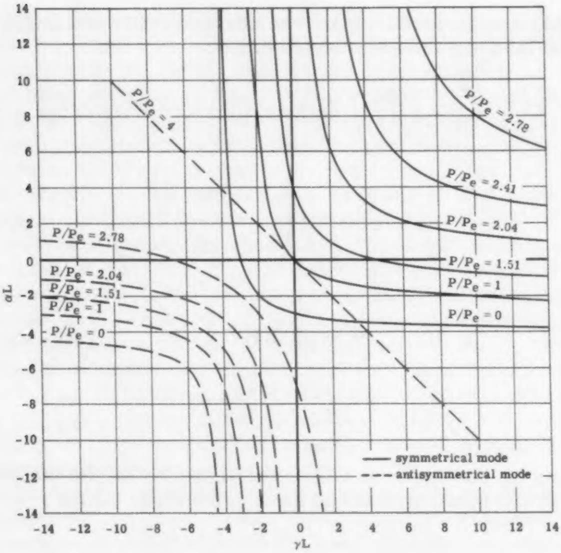

FIG. 6.—VARIATION OF BUCKLING LOAD WITH END-FIXITY

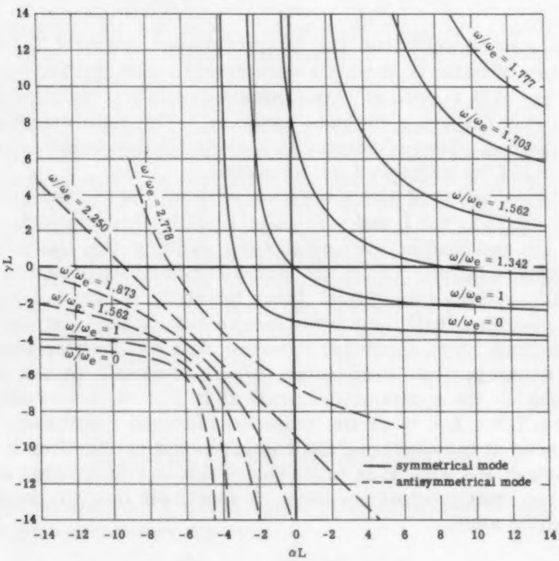

FIG. 7.—VARIATION OF FUNDAMENTAL FREQUENCY OF VIBRATION WITH END-FIXITY, P = 0

The curves of Fig. 6 may be approximated and put into an explicit form, giving the buckling load in terms of the end-fixities. The expression

$$\frac{P}{P_e} = \frac{4(\alpha L)(\gamma L) + 10.2\,(\alpha L + \gamma L) + 25}{(\alpha L)(\gamma L) + 5(\alpha L + \gamma L) + 25} \quad \ldots\ldots\ldots (9a)$$

gives the buckling load for any combination of end-fixities in the range of pinned-pinned to clamped-clamped with considerable accuracy. The largest deviation from the true buckling load is about 2%, and for the cases of pinned-pinned, pinned-clamped, and clamped-clamped, it is exact. An approximate expression previously proposed,[6] also gives the buckling load explicity in terms of the end-fixities, but with a lesser degree of accuracy. For negative end-fixities and for combinations of positive and negative end-fixity, the expression

$$\frac{P}{P_e} = \frac{(\alpha L)(\gamma L) + 4(\alpha L + \gamma L) + 12}{0.172(\alpha L)(\gamma L) + 1.955(\alpha L + \gamma L) + 12} \quad \ldots\ldots (9b)$$

should be used. This expression is also fairly accurate. It is exact for combinations of end-fixity that cause buckling without end load, and for combinations of end-fixity that produce buckling when the end load is the Euler load.

END-FIXITY AND VIBRATION

If instead of modifying Eq. 1 by the addition of an end compressive load, it is altered to take into account the inertia force generated by the mass of the beam in transverse motion, the well-known vibration equation

$$\frac{\partial^4 y}{\partial x^4} = \frac{\rho A}{EI} \; \frac{\partial^2 y}{\partial t^2} = 0 \quad \ldots\ldots\ldots (10)$$

is obtained, in which ρ is the mass density, that is specific weight divided by g. The solution of this equation for the condition of equal end-fixities has been discussed by the author.[4] For an arbitrary fixity at each end, the characteristic equation derived from Eq. 10 is

$$16\left(\frac{\beta L}{2}\right)^2 - 2(\alpha L + \gamma L)\frac{\beta L}{2}\left[\cot\frac{\beta L}{2} - \tan\frac{\beta L}{2} - \coth\frac{\beta L}{2} - \tanh\frac{\beta L}{2}\right] \ldots (11)$$

$$-(\alpha L)(\gamma L)\left[\cot\frac{\beta L}{2}\tanh\frac{\beta L}{2} - \tan\frac{\beta L}{2}\coth\frac{\beta L}{2}\right] = 0 \quad \ldots\ldots (12)$$

in which

$$\beta = \left[(\rho A \omega)/(EI)\right]^{1/4}$$

The characteristic roots, $\beta L/2$, are expressed in terms of the constants in Eq. 10 and the natural circular frequency of vibration, ω, as

$$\frac{\beta L}{2} = \frac{L}{2}\left(\frac{\rho A \omega^2}{EI}\right)^{1/4} \quad \ldots\ldots\ldots (13a)$$

6 "A Simple Approximate Formula for the Effective End-Fixity of Columns," by N. M. Newmark, Journal of Aeronautical Sciences, Vol. 16, 1949.

or

$$\frac{\omega}{\omega_e} = \left(\frac{\beta L}{\pi}\right)^2 \qquad \text{.............. (13b)}$$

For equal end-fixities it is found, as in the case of column instability, that the characteristic frequency equation is separable into equations that pertain to the symmetrical and antisymmetrical modes of vibration. Eq. 11 then factors into the symmetrical and antisymmetrical frequency equations given, respectively, as

$$\alpha L = \gamma L = -\frac{2\beta L}{\tan\frac{\beta L}{2} + \tanh\frac{\beta L}{2}} \qquad \text{....... (14a)}$$

and

$$\alpha L = \gamma L = \frac{2\beta L}{\cot\frac{\beta L}{2} - \coth\frac{\beta L}{2}} \qquad \text{....... (14b)}$$

Eq. 11 is plotted in Fig. 7. Note the strong resemblance to the buckling characteristic curves. The frequency characteristic curves are also hyperbolas and degenerate into the weightless beam characteristic curve when $\omega/\omega_e = 0$.

Fig. 7 shows that with increasing ω/ω_e, the separation between the constant frequency curves becomes larger, and, in the limit, as ω/ω_e reaches 2.25, the curve is simply a point at $\alpha L = \gamma L = \infty$. The characteristic curve of the corresponding antisymmetrical mode, with a frequency $\omega/\omega_e = 2.25$, is shown as a straight line. It is the only curve of the set in which a linear combination of end-fixities will result in a constant natural frequency of vibration. Note that this line does not go through the origin as the antisymmetrical buckling linear characteristic curve does. The general similarity in the pattern of the curves in Figs. 6 and 7 is an indication of the kinship between buckling and vibration. The transition that the variation in end-fixities produces in a frequency range of $\omega/\omega_e = 1$ to $\omega/\omega_e = 9/4$ resembles closely the transition that is produced in the buckling range of $P/P_e = 1$ to $P/P_e = 4$.

An approximate formula relating frequency of vibration to end-fixity in the range of pinned-pinned to clamped-clamped is

$$\frac{\omega}{\omega_e} = \frac{(\alpha L)(\gamma L) + 4(\alpha L + \gamma L) + 14}{0.4444(\alpha L)(\gamma L) + 2.560(\alpha L + \gamma L) + 14} \qquad \text{...... (15a)}$$

It gives the fundamental frequency of vibration explicitly, in terms of the end-fixities, with a maximum error of about 1%. When the ends are pinned-pinned, pinned-clamped, or clamped-clamped, Eq. 15a gives the exact frequencies. Another expression of this type, previously proposed,[7] gives frequencies with a maximum deviation of about 4% from the true frequency. For the case of negative end-fixities and combinations of positive and negative end-fixities the expression

$$\frac{\omega}{\omega_e} = \frac{(\alpha L)(\gamma L) + 4(\alpha L + \gamma L) + 12}{0.50(\alpha L)(\gamma L) + 2.56(\alpha L + \gamma L) + 12} \qquad \text{...... (15b)}$$

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[7] "A Simple Approximation of the Natural Frequencies of Partly Restrained Bars," by N. M. Newmark and A. S. Veletsos, Journal of Applied Mechanics, Vol. 19, 1952.

may be used. It is fairly accurate and is exact for combinations of end-fixity that produce zero frequency of vibration, and for combinations of end-fixity that give rise to a fundamental frequency of vibration equal to that of a pin-ended rod.

In plotting the variation of frequency of vibration with end-fixity in the manner shown in Fig. 7, it is difficult to indicate clearly the variation of the higher frequencies with end-fixity because of the overlapping curves that would result. This difficulty may be overcome by plotting frequency along one orthogonal axis and the end-fixity of one end of the member along the other axis, and specifying a constant or related end-fixity for the other end.[4] A field of parametric curves may be obtained in this manner showing the variation of frequency for the fundamental as well as the higher modes, with any combination of end-fixities.[8] Plots of this type are quite comprehensive, but they lack the descriptiveness of the curves shown in Fig. 7.

END-FIXITY OF VIBRATING COLUMNS

We now consider the composite problem of the effect of end-fixity and end load on vibration. The lateral force balance equation now becomes

$$\frac{\partial^4 y}{\partial x^4} + \frac{P}{EI}\frac{\partial^2 y}{\partial x^2} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \quad \ldots\ldots\ldots (16)$$

The displacement y is taken as $y = w \sin \omega t$ where w is the mode form and ω the harmonic circular frequency of vibration. Eq. 16 becomes

$$\frac{d^4 w}{dx^4} + p^2\frac{d^2 y}{dx^2} - \beta^4 y = 0 \quad \ldots\ldots\ldots\ldots (17)$$

From Eq. 17 one obtains the mode

$$w = A \sin \lambda_1 x + B \cos \lambda_1 x + C \sinh \lambda_2 x + D \cosh \lambda_2 x \quad \ldots (18)$$

in which λ_1 and λ_2 are the column vibration characteristic roots. They are expressed in terms of the column instability characteristic roots and the frequency characteristic roots as

$$\frac{\lambda_1 L}{2} = \left\{\frac{1}{2}\,\frac{pL}{2}^{\,2} + \left[\frac{1}{4}\left(\frac{pL}{2}\right)^4 + \left(\frac{\beta L}{2}\right)^4\right]^{1/2}\right\}^{1/2} \quad \ldots (19a)$$

and

$$\frac{\lambda_2 L}{2} = \left\{-\frac{1}{2}\left(\frac{pL}{2}\right)^2 + \left[\frac{1}{4}\left(\frac{pL}{2}\right)^4 + \left(\frac{\beta L}{2}\right)^4\right]^{1/2}\right\}^{1/2} \quad \ldots (19b)$$

8 "Bestimmung der Eigenschwingungszahlen von Durchlaufenden Tragern und Rahmen," by W. Mudrak, Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 28, 1948.

Boundary conditions 2 are now set into Eq. 18 to yield the characteristic column vibration equation

$$4\left[\left(\frac{\lambda_1 L}{2}\right)^2 + \left(\frac{\lambda_2 L}{2}\right)^2\right]^2 - \left[\left(\frac{\lambda_1 L}{2}\right)^2 + \left(\frac{\lambda_2 L}{2}\right)^2\right]\left[\alpha L + \gamma L\right]\left[\frac{\lambda_1 L}{2}\cot\frac{\lambda_1 L}{2} - \frac{\lambda_2 L}{2}\coth\frac{\lambda_2 L}{2} - \frac{\lambda_1 L}{2}\tan\frac{\lambda_1 L}{2} - \frac{\lambda_2 L}{2}\tanh\frac{\lambda_2 L}{2}\right] - (\alpha L)(\gamma L) \quad \ldots (20)$$

$$\left[\frac{\lambda_1 L}{2}\cot\frac{\lambda_1 L}{2} - \frac{\lambda_2 L}{2}\coth\frac{\lambda_2 L}{2}\right]\left[\frac{\lambda_1 L}{2}\tan\frac{\lambda_1 L}{2}\ \frac{\lambda_2 L}{2}\tanh\frac{\lambda_2 L}{2}\right] = 0$$

When the fixity at each end of the column is the same, Eq. 20 may be factored into two discrete characteristic equations that apply to column vibration in the symmetrical mode and column vibration in the antisymmetrical mode. They are, respectively

$$\alpha L = \gamma L = \frac{-2\left[\left(\frac{\lambda_1 L}{2}\right)^2 + \left(\frac{\lambda_2 L}{2}\right)^2\right]}{\frac{\lambda_1 L}{2}\tan\frac{\lambda_1 L}{2} + \frac{\lambda_2 L}{2}\tanh\frac{\lambda_2 L}{2}} \quad \ldots\ldots (21)$$

and

$$\alpha L = \gamma L = \frac{2\left[\left(\frac{\lambda_1 L}{2}\right)^2 + \left(\frac{\lambda_2 L}{2}\right)^2\right]}{\frac{\lambda_1 L}{2}\cot\frac{\lambda_1 L}{2} - \frac{\lambda_2 L}{2}\coth\frac{\lambda_2 L}{2}} \quad \ldots\ldots (21a)$$

Eq. 21 is plotted in Fig. 8 and Eq. 20 in Figs. 9 and 10. These curves show the variation of the square of the frequency.

$$\left[\left(\frac{\omega'}{\omega'_e}\right)^2 = \left(\frac{\beta L}{\pi}\right)^4\right] \quad \ldots\ldots\ldots\ldots (22a)$$

with the end compressive load

$$\left[\left(\frac{P}{P_e}\right) = \left(\frac{p L}{\pi}\right)^2\right] \quad \ldots\ldots\ldots\ldots (22b)$$

for fixed values of the end-fixity. By plotting the curves in this manner, there are obtained what appear to be sets of straight lines. The curve with the parameters $\alpha L = \gamma L = 0$, in Fig. 8, is actually a straight line and may be obtained from the well-known equation that relates the natural frequency of vibration of a pin-ended column to the column load. It is

$$\left(\frac{\omega'}{\omega'_e}\right)^2 = 1 - \frac{P}{P_e} \quad \ldots\ldots\ldots\ldots (23)$$

The other curves in this set are not quite linear, but the deviation from linearity does not become perceptible until quite large values of end-fixity are reached. The curves of Fig. 9, for the condition of one end fixed or one end

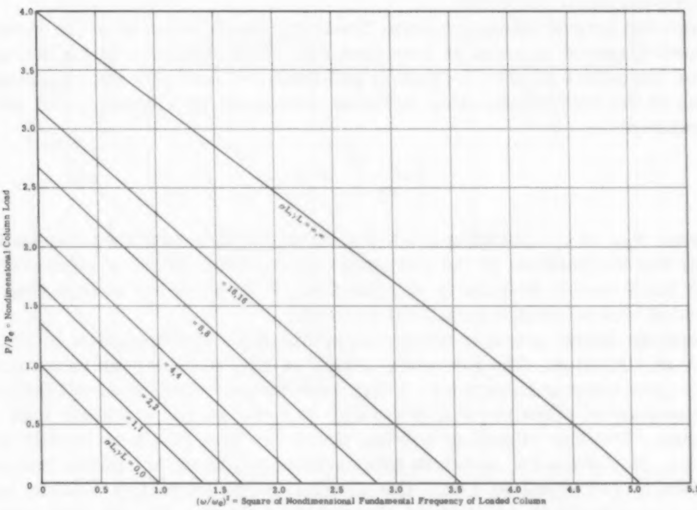

FIG. 8.—VARIATION OF FREQUENCY OF VIBRATION WITH COLUMN LOAD FOR THE CASE OF EQUAL END-FIXITIES

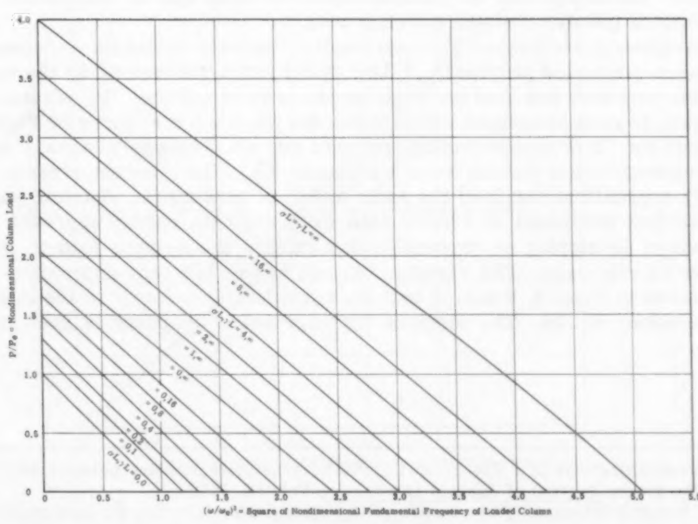

FIG. 9.—VARIATION OF FREQUENCY OF VIBRATION WITH COLUMN LOAD FOR THE CASE OF ONE END PINNED AND THE CASE OF ONE END CLAMPED

clamped, and the curves of Fig. 10 for mixed end-fixities also show an imperceptible deviation from linearity except when the end-fixity approaches the clamped-clamped condition.

Even the largest deviation from linearity, which is found in the curve for a clamped-clamped column, is less than 2%. This indicates that a straight line joining the points of zero frequency and zero end load is a very close approximation of the true relationship between frequency of vibration and end load. The expression

$$\left(\frac{\omega}{\omega_0}\right)^2 = 1 - \frac{P}{P_{cr}} \qquad (24)$$

in which P_{cr} is the buckling load of a column with a specified end-fixity, will permit the computation of the end compressive load, from a measured value of the fundamental frequency of vibration, with an error of less than 2% for any conditions of elastic rotational restraint.

Buckling loads are sometimes determined by extrapolation to zero frequency of vibration. The foregoing indicates that a linear extrapolation load should give very good results. It has been shown[2] that for small deflections, the frequency of vibration will always go to zero as the buckling load is approached. For the vibrating column problem, this result is readily deduced from Eq. 16. When β^4, which is proportional to the square of the frequency of vibration is set equal to zero, the equation reduces to the column equation whose characteristic solution is the buckling load. Eq. 24 also yields the buckling load for zero frequency, even though it is not an exact equation. For large amplitudes of vibration and axial restraint of the end supports, this relationship does not hold. It has been shown[9] that under these conditions the frequency of vibration does not go to zero as the buckling load is approached, and that vibrations can be obtained when the mean load on the column during vibration is greater than the buckling load.

The general problem of the interrelation between vibration and instability has been discussed previously.[3] One of the problems treated as the variation of frequency with end load for a clamped-clamped column. By strain energy methods, an area is formed within which the $\alpha L = \gamma L = \infty$ curve of Figs. 8 and 9 should lie. It is an interesting approach but not necessary when a straight line approximation entails an error of only 2%. The inherent error in strain energy approximations is of the same order of magnitude. Another paper on this subject published in 1936[10] also used a strain energy approach for the purpose of developing an expression that relates the column load to the frequency of vibration. The results obtained are rather poor approximations of the curves in Figs. 8, 9 and 10, and are considerably inferior to the linear approximation, Eq. 24. The form of the expression proposed is such that the

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9 "Free Vibrations of a Pin-Ended Column with Constant Distance between Pin Ends," by D. Burgreen, Journal of Applied Mechanics, Vol. 18, 1952.

10 "Natural Vibration Frequencies of Structural Members as an Indication of End Fixity and Magnitude of Stress," by B. C. Stephens, Journal of the Aeronautical Sciences, Vol. 4, 1936.

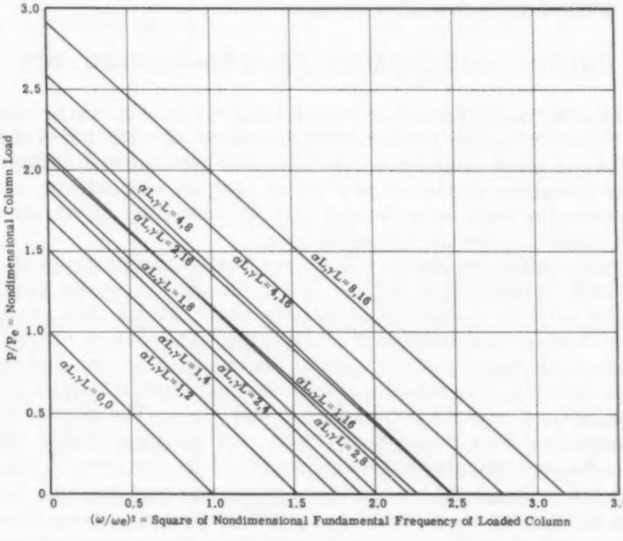

FIG. 10.—VARIATION OF FREQUENCY OF VIBRATION WITH COLUMN LOAD FOR MIXED END-FIXITIES

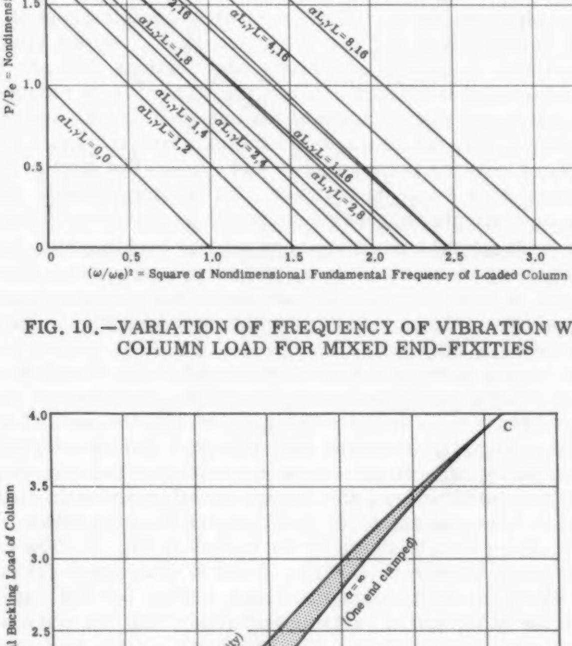

FIG. 11.—VARIATION OF BUCKLING LOAD WITH FREQUENCY OF VIBRATION OF UNLOADED COLUMN

frequency does not go to zero as the buckling load is reached. This discrepancy was pointed out[11] in a later article.

COLUMN LOAD CAPACITY FROM BEAM FREQUENCY

Consider now the possibility of determining the load carrying capacity of a column by measuring the fundamental frequency of vibration of the unloaded column. Both of these properties are dependent upon the end restraints. However, in the foregoing section it was shown that an approximate relationship exists between the load on a column and its frequency of vibration, without taking into consideration the end restraints.

Eq. 5 shows that there are an infinite number of combinations of end-fixity that will result in buckling at a given load, and Eq. 11 shows similarly that there are an infinite number of end-fixity combinations that are associated with a given fundamental frequency of vibration. There is no unique relationship between the frequency of vibration of the unloaded column and the buckling load, and to be sure, it is not possible to eliminate both αL and γL from Eqs. 5 and 11 to obtain a relation between $p L/2$ and $\beta L/2$. The similarity in form of these equations does suggest, however, that an approximate relationship may exist. An approximate expression proposed by Stephens[10] was obtained by plotting the fundamental beam frequency against buckling load for columns whose end-fixities were clamped-free, pinned-pinned, pinned-clamped, and clamped-clamped. A curve faired in through these four points thus gave graphically an approximate relationship between beam frequency and buckling load without consideration of end-fixity.

The fact that such a curve is not exact was pointed out,[10] and it was demonstrated that a sizable variation exists between the frequency of vibration of a pinned-clamped beam and a beam with equal end-fixities, both of which have the same buckling load. In selecting this example, perhaps fortuitously, to show the impossibility of a single curve representing the variation between beam frequency and buckling load, the largest possible deviation in frequency for a given column load was obtained. The true relationship between the beam frequency and buckling load is shown by the curves in Fig. 11. The curves are drawnf from the axes intercepts in Figs. 8 and 9. The upper curve is drawn for the case of equal end-fixities and the lower curves for the cases of; one end pinned and the other end of variable end-fixity; and one end clamped and the other end of variable end-fixity. Single curves of this type can only be drawn by specifying either a relationship between αL and γL or fixing αL or γL. The curves shown in Fig. 11 represent the envelope of all possible curves that show the variation of beam frequency with buckling load. Corresponding values of beam frequency and buckling therefore fall inside the shaded area regardless of end-fixity. Note that the envelope narrows in the vicinity of the pinned-pinned and clamped-clamped conditions. It is at the pinned-clamped point that the envelope is widest and could show the largest spread in either beam frequency or buckling load with a change in fixity. The curve[10] proposed originally is a smooth curve passing through points A, B, C. For use in estimating the buckling load from a frequency measurement, it is on the conservative side, because it always gives lower than actual buckling loads.

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11 "Effective End Restraint of Columns by Frequency Measurements," by H. Lurie, Journal of the Aeronautical Sciences, August, 1951.

If points A and C are joined by a straight line, it is found to represent quite well the frequency-buckling relationship for columns with equal end-fixity. The linear expression is

$$\frac{P}{P_e} = 2.4 \frac{\omega}{\omega_e} - 1.4 \qquad \text{. . . . .} \qquad (25)$$

For equal end-fixities, the maximum error is about 2%. For other combinations of end-fixity, the approximation is not as good.

SUMMARY

The study of the characteristic equation of a weightless beam shows that instability can be obtained as a result of negative end-fixity alone. The practical significance of negative end-fixity is demonstrated in the examination of the instability of some type of frames.

The general effect of end-fixity, including negative end-fixity, is studied for the vibrating beam and the column. The similarities in these two systems is pointed out. Approximate expressions are derived which give explicitly the buckling load and fundamental frequency of vibration in terms of the end-fixity.

The question of the degree of accuracy in the assumption,that the square of the fundamental frequency of vibration varies linearly with end compressive load, for all combinations of end-fixity, is resolved by solving the characteristic equation of a vibrating column and plotting the results. The indication is that the maximum error, entailed in the assumption that the square of the fundamental frequency varies linearly with the column load, is about 2%.

The problem of estimating the buckling load of a column, with arbitrary end-fixities, by measuring the fundamental frequency of the column in the unloaded state, is examined. Although there is no unique relationship between the buckling load of a column and the fundamental frequency of the unloaded column, it is demonstrated that an approximate correspondence between the buckling load and vibration frequency of the unloaded column does exist. An expression relating these two quantities has been suggested.

APPENDIX · NOTATION

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y	lateral displacement of beam or column
x	axial distance measured from enter of member
L	length of beam or column
α	end-fixity at left end of member = $\left(\frac{d^2y}{dx^2}\right) / \left(\frac{dy}{dx}\right)$ at $x = -L/2$
γ	end-fixity at right end of member = $-\left(\frac{d^2y}{dx^2}\right) / \left(\frac{dy}{dx}\right)$ at $x = L/2$

ω	circular frequency of vibration of beam or column
ω_e	circular fundamental frequency of vibration of a pin-ended beam
ω_0	circular fundamental frequency of vibration of a beam with a specified end-fixity
P	compressive axial end load
P_e	buckling load of a pin-ended column
P_{cr}	buckling load of a column with a specified end-fixity
E	modulus of elasticity
I	cross section rectangular moment of inertia
p	$(P/EI)^{1/2}$
ρ	mass density = specific weight divided by g
A	cross-section area
t	time
β	$\left[(\rho A\omega^2)/(EI)\right]^{1/4}$

$$\lambda_1 L/2 = \left\{(1/2)(pL/2)^2 + \left[(1/4)(pL/2)^4 = (\beta L/2)^4\right]^{1/2}\right\}^{1/2}$$

$$\lambda_2 L/2 = \left\{-(1/2)(pL/2)^2 + \left[(1/4)(pL/2)^4 + (\beta L/2)^4\right]^{1/2}\right\}^{1/2}$$

Journal of the
ENGINEERING MECHANICS DIVISION
Proceedings of the American Society of Civil Engineers

PHYSICAL METALLURGY AND MECHANICAL PROPERTIES OF MATERIALS: BRITTLE FRACTURE

By B. L. Averbach[1]

FOREWORD

The Engineering Mechanics Divisions Committee on Mechanical Properties of Materials conceived of the Symposium on Physical Metallurgy and Mechanical Properties of Materials as a means of summarizing, for the civil engineering profession, the current state of knowledge and some of the most recent developments in the understanding of materials behavior. The subjects of ductility, creep, brittle fracture, and fatigue have become of increasing concern in engineering applications, to the point where phenomenological descriptions of these aspects of materials behavior are becoming less than adequate for our needs.

The science of materials, stemming from physical chemistry, solid state physics, physical metallurgy, and mechanics of solids, gives us qualitative descriptions of the fundamental processes involved in the flow and fracture of solids. These fundamental concepts were presented by distinguished authorities in the 1956 Symposium on Physics of Engineering Materials organized by J. L. Waling and held at Pittsburgh, Pa. Frederick Seitz discussed imperfections in crystals, Thornton Reed, Jr., discussed dislocations, Clarence Zener discussed internal friction in metals, E. P. Blizzard and A. M. Weinberg discussed materials for radiation shielding, and G. J. Dienes discussed the effects of radiation on materials properties. The implications of these subjects in civil engineering practice were summarized by Glenn Murphy. Much of the

Note.—Discussion open until May 1, 1961. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. EM 6, December, 1960.

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material presented has been published in the literature of their own fields and may be found in their published works. The applications of such fundamental concepts have brought about entirely new materials as well as improvement in traditional materials and indeed have created entirely new industries.

Theoretical relationships of the quantitative kind desired for engineering design purposes are, in many cases, still in the state of development. The development of such fundamental theory as exemplified by the study of brittle fracture by B. L. Averbach ("Brittle Fracture," Proceedings Paper, 2686) and the relaxation theory of creep of metals by F. H. Ree, T. Ree, and H. Eyring ("Relaxation Theory of Creep of Metals," Proceedings Paper 2333, Journal of the Engineering Mechanics Division, January, 1960, p. 41) and the importance of these concepts to engineering practice as indicated in the discussions of ductility by J. M. Frankland ("Ductility and the Strength of Metallic Structures," Proc. Paper 2687), of fatigue in structural materials by H. J. Grover ("Fatigue of Structural Materials," Proc. Paper 2688) and in the review of applications in civil engineering by Glenn Murphy ("Metallurgical Advances and Civil Engineering," Proc. Paper 2689) are considered by the committee to be invaluable to the profession of civil engineering.

In the present "Age of Materials," the problems of providing the materials suitable for the anticipated service conditions of many engineering applications require that all engineers be familiar with the latest and best of information pertaining to materials behavior. The practicing engineer must be cognizant of the rapidly growing field of materials science, which is so significant that, in some cases, it is causing large scale revisions of engineering college curricula.

The Committee on Mechanical Properties of Materials, through the 1956 Symposium on Physics of Engineering Materials, the 1958 Symposium on Physical Metallurgy and Mechanical Properties of Materials, the 1959 Symposium on the Physico-Chemical Nature of Soils, the 1960 Symposium on Nondestructive Testing of Materials, and the Symposium on Teaching of Materials in Civil Engineering Curricula being planned for 1961, have attempted to incorporate within the literature of the American Society of Civil Engineers a significant amount of the new knowledge in the field of materials behavior. Many of the participants in these symposia have been from professions other than civil engineering. A sincere expression of appreciation is hereby tendered to all who have contributed to this effort.

Joseph F. Throop, Chairman, E.M.D.
Committee on Mechanical Properties
of Materials, ASCE

SYNOPSIS

Recent experimental work on brittle cleavage fractures in iron and steel has shown that these failures originate with cleavage microcracks. Microcracks with lengths of one grain diameter and longer have been observed in tensile specimens that have been unloaded just below the fracture stress. It has also been shown that local plastic flow of the order of the yield strain occurs prior to the formation of these microcracks even though the subsequent failure involves

very little overall deformation. Some of the current dislocation theories predict the formation of microcracks of this type, and the correlation between theory and experiment is encouraging.

INTRODUCTION

A brittle fracture may be defined as one which absorbs a relatively small amount of energy in propagation. Failure by the cleavage of a large fraction of the individual grains along (100) planes is a form of brittle fracture that occurs in the common structural steels, and this problem must be taken into account in the design of large continuous structures. Brittle cracks propagate through steel plates with about one-third the velocity of sound and with very little plastic deformation on a macroscopic scale. The fracture surfaces have a distinct chevron pattern, pointing back to the source of the failure, which is almost always a stress concentration in the form of a sharp notch introduced during construction, a sharp corner inherent in the design, or a crack produced by metallurgical damage in construction or service. Brittle cracks stop when the residual strain energy is no longer sufficient for propagation or when a particularly tough plate or structural feature is encountered. It has been shown in many cases that localized yielding is associated with the start of a running crack, and there is evidence of plastic deformation at the crack interfaces. The structure as a whole does not have to yield prior to failure, and the danger of a catastrophic failure increases as the size of the structure increases and as the service temperature is lowered.

Several procedures have been used to minimize the danger of brittle fracture. Sharp corners and other notches in the design are eliminated. An outstanding example of this approach occurred in World War II cargo ships in which the incidence of brittle failure was greatly reduced by redesigning a troublesome hatch corner. Another approach involves the careful inspection during construction in order to reduce the introduction of flaws. This procedure has been particularly effective in pipelines. These measures have not eliminated the problem, however, and it is recognized that the last safety factor must be built into the steel. Considerable progress has been made in the understanding of what can be done to improve steels in this respect, but the mechanism of fracture on an atomic scale is not well understood, and the present day precautions may be only temporary measures on the way to a better solution.

Descriptions of service failures in ships, (1), (2), (3),[2] tanks, and other structures (4) and large generator rotors (5) are available and will not be discussed here. General design criteria have also been summarized recently (6). The engineering tests that have been used to investigate brittle fracture in steels have also been summarized, (3), (7) but a few of these are discussed in the next section in order to indicate the approaches that have evolved in the specification of steels to resist brittle failure. Recent experimental work (as of 1960), on the mechanisms of cleavage failure and some of the current theories of brittle fracture, are presented in the final section.

It should be noted that brittle failures do not necessarily have to occur by cleavage. For example, steels and other alloys that have been heat treated to

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[2] Numerals in parenthesis—thus; (1)—refer to corresponding items in the Appendix Bibliography.

very high yield strengths are capable of only very limited plastic flow. High applied and residual stresses may combine to produce the proper conditions for the nucleation of microcracks which can propagate with very low energy absorption because of the low ductility. A cleavage mechanism may not be involved in these failures, but little is known about the mechanisms of these fractures, and this type of high speed tearing phenomenon will not be discussed. Materials of this type are finding increased use in thin skill construction, and it is evident that a concerted research effort in this area is required.

ENGINEERING TESTS AND SPECIFICATIONS

The appearance of the fracture surfaces in service cases of brittle failure has been well documented (3), (7). A chevron pattern that points back to the origin of failure is frequently observed in failures occurring in mild steel, but this is not a characteristic of brittle steel failures alone since it has also been observed in glass and plastics. The chevron pattern appears to be associated with a crack that proceeds in a discontinuous fashion, with a sequence of initiations ahead of the main crack front followed by a breakdown of the intervening material to form a union with the principal crack. The crack front is also not straight, with the main fracture front extending in the center well in advance of the surface trace of the crack. In mild steel it appears that cleavage microcracks are initiated ahead of the main crack and it is this repeated initiation of microcracks that provides the discontinuous feature of the traveling crack. The problems of initiation and propagation of cleavage cracks thus tend to merge since conditions for the initiation of a cleavage microcrack may be very close to those required for the discontinuous propagation of a gross crack.

One of the engineering approaches to this problem has been concerned with the search for tests that produce brittle failure and the correlation of the test data with service behavior. Some of these tests appear to emphasize the initiation of cleavage cracks, others the stopping of a propagating brittle crack, but almost all of the tests involve the introduction of a notch and the observation of the onset of brittle behavior as the test temperature is lowered. Many tests also try to reproduce the dynamic aspects of a traveling crack, and this feature may be quite important in correlations with service behavior because the yield point of steel is very much dependent on the strain rate. Each of these tests emphasizes different features of the brittle-fracture phenomenon, and it is not surprising that they frequently differ in their evaluation of the ability of a material to resist brittle failure.

The tests almost always define a transition temperature, below which brittle fracture occurs under the test conditions and these critical temperatures are used (1) in direct comparison with the temperature for service failure of the same material and; (2) to evaluate the effectiveness of various metallurgical variables in lowering the transition temperature. It is now generally agreed that a decrease in transition temperature in any of the tests probably reflects an improvement in the material even though not all of the tests may indicate an equal improvement.

The engineering tests have been summarized recently (3), (6), (7), but the V-notch Charpy impact test deserves special mention because of its widespread use. A typical set of impact data are shown in Fig. 1 for specimens taken from a 3/4 in. thick plate of rimmed steel. The energy absorption changes considerably over a rather small temperature interval and the fracture appearance changes correspondingly from the fibrous appearance associated with the high

energy region to the specular appearance resulting from the cleavage of many grains along (100) crystallographic planes. The temperature at which 15 ft-lbs is absorbed is frequently defined as a ductility transition temperature T_d and is associated with the temperature region in which cleavage cracks initiate the brittle fracture.

The 15 ft-lb transition temperature is based on an investigation of the plates in World War II cargo ships that exhibited brittle fracture. It was shown thai all of the plates wherein brittle fractures started absorbed less than 10 ft-lbs at the failure temperature (1), (2), whereas the end plates consistently absorbec more energy. It should be emphasized that this correlation refers to the semi-killed and rimmed steels involved in that particular ship construction, and subsequent data have indicated that the energy level for the ductility transition may have to be raised to about 20 ft-lb for killed steels and some alloy grades.

Another approach to the V-notch Charpy test defines a fracture transitio temperature T_f in terms of the fracture appearance. The temperature for 50 fibrous appearance corresponds to a condition in which brittle fracture may be

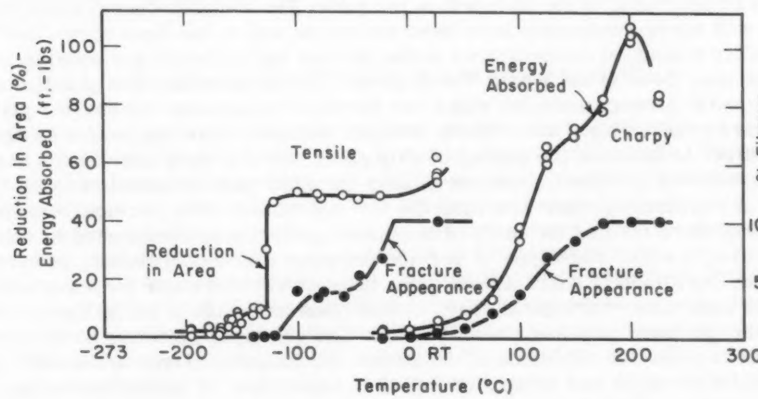

FIG. 1.—TENSILE AND V-NOTCH CHARPY IMPACT TRANSITION OF LARGE GRAIN (ASTM GS NO. 4) PROJECT STEEL E 0.22C, 0.36 Mn, 0.002 Si

initiated by a ductile crack. This situation might occur in a structure which already contained a crack which was extended because of high local stresses at the tip. As long as the crack proceeds by shear a large amount of local deformation would be required (in mild steel), considerable energy would be needed and the crack would move slowly. When the crack reaches a critical size, depending on the temperature, the stress field in front of the crack could become large enough to initiate cleavage microcracks that could join the original shear crack by cleaving and tearing the intermediate material. Conversely, at a given temperature, if the material were above the fracture transition temperature and were presented with a running brittle crack, it could stop the crack because of the high energy required to shear a large portion of the material. It is apparent that the fracture transition T_f is higher than the ductility transition T_d.

The anticipated service conditions are quite important in considering which transition temperature to consider. For example, in the case of a submarine hull

it might be expected that severe local deformation could result in a shear crack. It would be important to prevent this shear crack from turning into a rapidly propagating cleavage crack, and the fracture transition would thus be of prime importance. Similarly, a large generator rotor might contain internal flaws that remain undetected because of the massive size of the forging. One of these cracks could reach the critical size during service or overspeed tests and eventually propagate as a brittle crack. On the other hand, a storage vessel might be constructed under conditions in which there was sufficient inspection to avoid the introduction of major flaws. At low ambient temperatures, however, it would be necessary to avoid the initiation of a cleavage crack in the vicinity of a stress concentration, and the ductility transition temperature could be used as a basis for choosing the material.

Both philosophies are used in merchant ship building. The specifications of Lloyds Register of Shipping (8) now include a class of plate that requires 35 ft-lb and a 30% fibrous appearance at 0°C. This specification was chosen on the basis of service data for plates which successfully stopped running brittle cracks, and these plates are specified for key regions in the ship. American practice specifies the composition and steel making practice of all of the material in the ship according to the thickness of the plate. The V-notch Charpy characteristics of current ship plate have been monitored, and it has been shown that the ductility transition temperatures of the present day material are considerably lower than those of the World War II plates. The composition and practice are designed to produce material with a low ductility transition in the heavier plates that are presumably at the regions of higher stress. There has been a continuing effort to improve the quality of ship plate, and the work carried out under the sponsorship of the Committee of Ship Steel has been reviewed recently (9).

The engineering tests are valuable for correlation with service behavior, and they have formed the basis of the investigative procedures used to select improved steels. Each type of service requires a new correlation, however, and the multitude of correlations using tests of various kinds for a variety of steels used under variegated service conditions has made it desirable to pursue this problem on a fundamental basis. The principal objective of the basic research is an understanding of the atomic mechanisms of fracture so that the resultant concepts can be applied to a broad spectrum of materials and problems. Several recent theoretical and experimental approaches will be discussed.

METALLURGICAL FACTORS IN BRITTLE BEHAVIOR

Many of the steelmaking variables have been evaluated for the weldable mild structural steels in terms of impact transition temperatures. It has been shown that the transition temperature is lowered as the carbon content is lowered and the manganese content is increased in structural steels, (3) and several low carbon-high manganese steels have been introduced because of their improved resistance to brittle failure. It is significant to note that alloy additions other than manganese either raise or have no effect on the transition temperatures of these steels Aluminum-treated steels made to fine-grain practice are superior to rimming or semi-killed grades, and the improvement may be further enhanced by normalizing. Investigations have shown that the transition temperature is raised as the ferrite grain size is increased, and there is evidence to indicate that the transition temperature for a steel cooled slowly from the annealing temperature is higher than for a steel that has been rapidly cooled (10).

There is a limit, however, to the improvement that may be attained in ferritic steels by these means, and the basic mechanisms by which these improvements are attained are not well understood. Furthermore, these empirical concepts do not extrapolate readily to higher alloy steels or to other types of microstructure such as bainite or martensite. Experience with the coarse bainitic microstructures which have been used in some large generator rotors has indicated that they are susceptible to brittle cleavage failure despite the high alloy content of the steel. On the other hand, fine bainitic or martensitic microstructures of the same composition may have rather low transition temperatures, and it is probable that the improvement may be associated with the small size of the ferrite regions in the latter microstructures. The metallurgical factors at play in the brittle behavior of the high strength steels have not been investigated in the same detail as the weldable ferritic grades, and it seems as if these data will now be required in increasing quantity for some of the new structural applications. One of the characteristic features of brittle cleavage failure is that it does not occur in metals with face-centered cubic crystal structures. Thus, the austenitic stainless steels do not exhibit brittle failure, and this material is frequently used for structures that must operate at temperatures below -80°C.

ATOMIC MECHANISMS OF CLEAVAGE FRACTURE

Dislocation concepts are used in most of the current theories of brittle fracture. These theories have been reviewed recently (11), (12), (13), and the principal features of these theories will be discussed together with some recent pertinent experimental work. Most of the fundamental experimental work uses unnotched bend or tensile specimens in order to simplify the stress pattern, and the details of fracture are studies at low temperatures. Fig. 1 shows that the tensile transition temperatures are considerably lower than the Charpy V-notch impact transitions, but that the principal features of a loss in ductility and a change in the fracture appearance are common to both tests. The influence of notch and impact loading on the transition temperature are quite evident.

The relationship of the ductility and fracture parameters to the tensile properties is shown in Fig. 2. Several modes of fracture are observed at temperatures below 20°C. At temperatures down to about 0°C, the fracture is entirely ductile with considerable necking prior to fracture. Between 0° and -120°C, a ductile crack starts at the center of the specimen and propagates to a fraction of the diameter with the remainder of the fracture occurring by cleavage. The size of the ductile crack required to initiate the cleavage failure becomes smaller as the temperature is lowered (Fig. 3), but the overall ductility is still quite high because necking is required to initiate the ductile crack. This temperature range corresponds to the fracture transition. At temperatures below -120°C the mode of fracture changes. The first observable crack is a cleavage microcrack of the order of one grain diameter in size, and the final failure has the gross appearance of 100% cleavage. It is important to note that mechanism of initiation changes completely in this region, and -120°C corresponds to our previous definition of a ductility transition.

It should be noted in Fig. 2 that the lower yield stress is reached prior to fractures that are classed as entirely brittle. In addition, a well defined elastic limit, corresponding to an observable plastic strain of 1×10^{-6}, is observed in all cases prior to the discontinuous yield. A reduction in area is observed at temperatures down to about -170°C even though the fracture appearance is

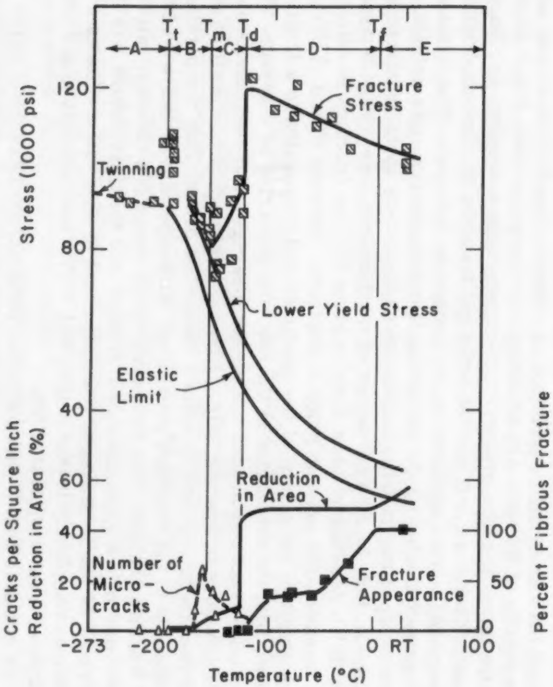

FIG. 2.—RELATION OF TENSILE PROPERTIES TO FRACTURE APPEARANCE AND MICRO-CRACKS FOR COARSE GRAIN (ASTM GS NO. 4) PROJECT STEEL E

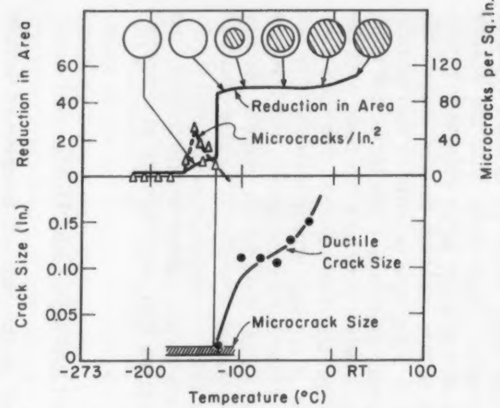

FIG. 3.—FRACTURE OF COARSE GRAIN (ASTM GS NO. 4) PROJECT STEEL E

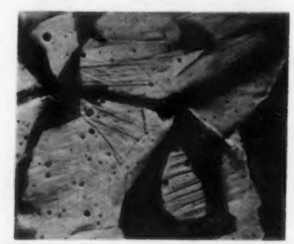

FIG. 4.—MICROCRACK AT SURFACE OF UNFRACTURED E STEEL SPECIMEN. 250X

entirely cleavage, and an examination of the crack surfaces and the microcracks shows that considerable local plastic deformation is associated with the ends of microcracks and with the crack interfaces. A typical microcrack is shown in Fig. 4, showing the surface of a cylindrical specimen which was electropolished prior to testing and unloaded prior to fracture after stressing to the yield point. The cleavage microcrack and the attendant distortion are quite evident.

The relationship between cleavage microcracks and discontinuous yielding was demonstrated by W. S. Owen et al (14) in an experiment in which Lüders bands were passed along thin flat prepolished specimens of this same steel at liquid nitrogen temperature. The Lüders bands were passed only part way along the specimen, and the specimen was unloaded and examined for microcracks. The microcracks were very similar in appearance to that shown in Fig. 4, and the location of the cracks is shown schematically in Fig. 5. The experiment was repeated recently with an additional variation: the specimen

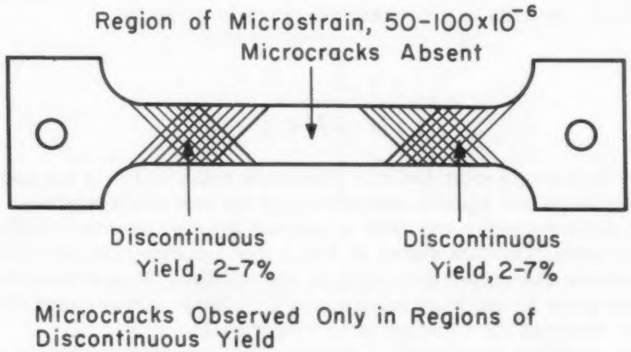

FIG. 5.—LOCATION OF MICROCRACKS IN FLAT TENSILE SPECIMEN

was heated to room temperature after unloading and given an additional tensile elongation in order to reveal microcracks that might have closed partially on unloading. The results were the same in both cases. Microcracks were observed only in the region that had yielded discontinuously. They were not observed in the region between the Lüders band, even though this region had suffered a plastic strain of 50-100 x 10^{-6}. It is thus evident that the cleavage microcracks require a local deformation of the order of the discontinuous yield prior to initiation, but the overall extension of the specimen may be very small if the Lüders band does not travel far before the specimen fractures. Cleavage fracture is thus closely associated with yielding, and the same factors that influence the yield point would be expected to influence the cleavage fracture stress. This has been generally observed.

Several general experimental features must be considered by the theoretical treatments of brittle cleavage failure.

1. Cleavage failure occurs in body-centered cubic and some hexagonal crystal structures but not in face-centered cubic materials. The tendency for cleavage failure is greatly influenced by the presence of interstitial elements such as carbon and nitrogen.

2. Plastic flow precedes the cleavage crack, and the surface of the interface exhibit considerable local deformation.

3. The tendency toward brittle fracture is accentuated at low temperatures. The yield strength of iron and steel rises sharply with decreasing temperature below 20°C and the brittle behavior is undoubtedly associated with this rise in yield strength. Face centered cubic metals show only a small rise in yield with decreasing temperature and are not subject to cleavage failure.

4. Cleavage failure is favored by high strain rates. Recent work (15) (1949) has indicated that the discontinuous yield in steel does not occur instantaneously on application of the load, but there is a delay time which depends strongly on the temperature, varying from about 10^{-4} seconds at room temperature to about 10^3 seconds at liquid nitrogen. A high strain rate can thus raise the yield stress in the same way as a decrease in temperature.

One of the early theoretical approaches to the problem of brittle failure was made by Griffiths who considered the stress required to propagate a sharp crack in an elastic medium. For a two-dimensional crack of length 2C the tensile stress σ required to propagate the crack is given by:

$$\sigma = \left[\frac{2\,E\,\gamma}{\pi\left(1 - \nu^2\right) C}\right]^{1/2} \qquad \text{(1)}$$

in which E is Young's modulus, ν is Poisson's ratio, and γ is the specific surface energy (ergs per square centimeter) of the new crack surface. Cracks of the order of 10^{-4} cm are required to account for the observed brittle fracture strength of steel. The data shown in Fig. 2 also indicate that one of the primary assumptions associated with Eq. 1 is not fulfilled. It is evident that plastic flow occurs prior to cleavage failure, and the elastic stress concentration factor used in deriving the equation is no longer valid.

Orowan modified the Griffiths' equation by recognizing that considerable plastic work is associated with the passage of a cleavage crack in ferritic steel. This consideration leads to the equation

$$\sigma = \left[\frac{4\,E\,p}{\pi\,d}\right]^{1/2} \qquad \text{(2)}$$

in which p is now the effective surface energy, including the plastic work. This effective surface energy has been estimated to be about 10^6 ergs per cm^2. This concept leads to the presence of cleavage microcracks of the order of the grain size, d. The presence of such microcracks prior to fracture has been demonstrated by J. R. Low, (16) Owen, et al (14). A typical example is seen in Fig. 4. The principal difficulty is in accounting for the temperature dependence, and it must be assumed that the term p varies strongly with temperature. In addition, there is little indication in this picture for the need for plastic flow prior to cleavage. It would appear that this approach accounts for some of the macroscopic features of the phenomenon and represents only a part of the picture.

A. N. Stroh (11) introduced the concept of a dislocation pile-up in a slip band as supplying the conditions for the initiation of a cleavage crack. C. Zener (17) had considered earlier that a blocked ship line would resemble a freely slipping crack under a shear stress, and Stroh considered the dislocation pile-up as producing a stress concentration equivalent to that of a Griffith's crack.

Such a dislocation pile-up along a slip line against a grain boundary barrier (Fig. 6) could produce a crack along a cleavage plane. The condition for the formation of a two-dimensional crack can be derived simply by considering that all of the strain energy associated with the dislocations is relieved by the formation of the crack. This gives, directly,

$$\sigma_s \, n \, b = 2 \, \gamma \qquad (3)$$

in which σ_s is the applied shear stress, n the number of dislocations in the pile-up, b the Burgers vector of the dislocation, and γ the specific surface energy of the crack. This equation is similar to Stroh's with the exception of the numerical factor 2, which is $\frac{3}{8}\pi^2$ in the more exact computation.

The influence of grain size may be seen in the following way: the dislocation pile-up is considered to occur over a region d/4, and the local shear strain is

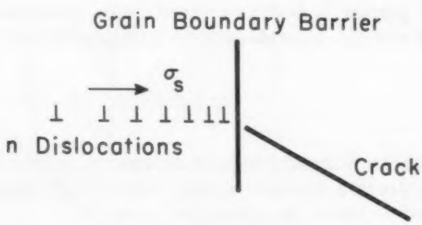

FIG. 6.—DISLOCATION PILE-UP RELIEVED BY A CRACK (AFTER STROH)

thus given by $\epsilon_s = 4 \, n \, b/d$. The shear strain is given approximately by σ_s/G, in which G is the shear modulus. Introducing these into Eq. 3 yields

$$\sigma_f \approx 2 \, \sigma_s = K \, d^{-1/2} \qquad (4a)$$

in which σ_f is the fracture stress and $K = (32 \, \gamma \, G)^{1/2}$. (Stroh's more exact computation gives $K = \left[6 \, \pi \, \gamma \, G/(1 - \nu)\right]^{1/2}$). In order to correlate with the observed dependence of the fracture stress on the grain size a frictional stress, σ_0 is introduced to give the final equation

$$\sigma_f = K \, d^{-1/2} + \sigma_0 \qquad (4b)$$

This concept has several difficulties. The cleavage crack is pictured as being produced by shear stress alone, but it is evident from other data that the presence of normal tensile stresses is very effective in raising the cleavage transition. In order for the dislocation pile-up to act as a Griffiths crack it is necessary that the surrounding grains do not flow. This implies that all of the dislocation generators near the pile-up must be pinned by interstitial atoms so that slip is prevented and cleavage occurs. The dislocation pinning mechanism is strongly temperature dependent and this would probably provide the proper

temperature dependence except that it is difficult to see why slip instead of cleavage does not occur in the surrounding grains. In fact, an expression of the form or Eq. 4 describes the influence of grain size on yield stress, σy, with greater accuracy than it does on the fracture stress.

Fig. 2 shows that the yield and fracture stresses in the 100% cleavage region have very similar values. Finite cleavage microcracks of the order of a grain diameter are formed at the yield stress and the frequency with which these cracks occur is plotted in Fig. 2. Thus, the observed fracture stress is for a material containing numerous microcracks, and it would appear to be of doubtful fundamental significance. In addition, the Lüders band experiments of Owen, et al. (14) show that the microcracks occur only in the region of discontinuous deformation in which the plastic flow is very heterogeneous and much more complex than the simple pile-up shown in Fig. 7.

A. H. Cottrell (12) has recently (1958) reconsidered the problems associated with dislocation pile-ups. He proposes that dislocations travelling on two intersecting (110) slip bands could produce a dislocation pile-up in the form of a giant dislocation with a Burgers vector nb. This could act as an atomic knife along (100) cleavage planes in body-centered cubic material. Such a pile-up is shown in Fig. 7, and energy considerations similar to that used in Eq. 3 yield

$$\sigma_f \, n \, b = 2 \, p \quad \dots\dots\dots\dots\dots\dots (5)$$

It should be noted that the normal fracture stress, σ_f, enters here instead of the shear stress. The effective surface energy term, p, includes the irreversible plastic work required to form the cleavage crack.

The grain size dependence is obtained as follows: it is assumed that the fracture stress σ_f is approximately equal to the yield stress, σ_y, and that the yield stress has a grain size dependence of the form,

$$\sigma_y = K_y \, d^{-1/2} + \sigma_{y\,o} \quad \dots\dots\dots\dots\dots\dots (6)$$

in which σ_{yo} is the frictional component of the yield stress. The shear stress at yield is $\sigma_s = \sigma_y/2$. The dislocation displacement, n b, is then given approximately by,

$$n \, b = \left(\sigma_s - \sigma_i\right) d/G \quad \dots\dots\dots\dots\dots\dots (7)$$

in which σ_s is the applied shear stress and σ_i is the frictional component of the shear stress. Introducing these relationships into Eq. 5 yields

$$\sigma_s \, K_y \, d^{1/2} = \beta G \, p \quad \dots\dots\dots\dots\dots\dots (8)$$

The factor β is approximately unity in the tension test. A comparison with experiment gives a value of 18,000 ergs per cm^2 for p. This is about 10 times the value of the surface free energy, γ.

Eq. 8 is applied in an interesting way. If the left-hand side is smaller than the right-hand side there is insufficient energy to propagate a cleavage crack beyond a certain length. In this case, a cleavage microcrack may form but it will not grow. When the left-hand side is larger than the right, the yield stress

is high enough to make the cleavage crack grow to complete failure. A transition point is thus defined, and Cottrell has assumed that this is the ductility transition, T_d. The largest stable microcrack may be computed from the Cottrell equations, and this turns out to be:

$$C = \frac{d}{2\pi(1-\nu)} \frac{(S-1)}{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

in which $S = \sigma_s/\sigma_i$. The factor S varies with grain size, but a typical value for steel with $d^{-12} = 6$ mm$^{-1/2}$ is S = 3. Taking $\nu = 1/3$ this gives C = d/6, or C $\approx$d. Fig. 4 shows an example of such a microcrack.

Fig. 2 shows, however, that the transition temperature described by Eqs. 5 and 8 may refer to a temperature somewhat lower than the ductility transition temperature T_d. The temperature regions wherein different fracture mechanisms operate are shown in the figure. In region E, above the fracture transition T_f, the fracture is entirely by shear. In region D between T_f and T_d, the

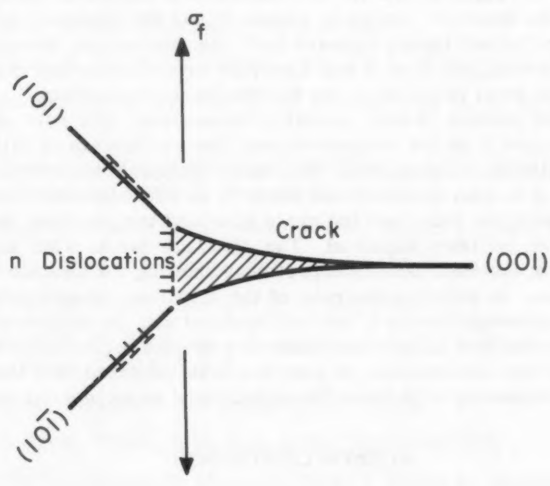

FIG. 7.—CLEAVAGE CRACK FORMED BY COALESCENCE OF DISLOCATIONS ON INTERSECTING SLIP PLANES. (AFTER COTTRELL)

fracture is started by a shear crack and is completed by cleavage. In this region, the size of the shear crack required to initiate a cleavage failure decreases as the temperature decreases (Fig. 3).

In region C, which is between T_d and T_m, we propose that the failure is initiated by cleavage microcracks, and that these microcracks join up to form the final fracture. The microcrack curve shows the number of stable microcracks prior to fracture in this region, and it is evident that a sharp maximum is reached at T_m, which we shall call the microcrack transition temperature. T_m corresponds also to the sharp valley in the fracture stress at which the yield stress σ_y and the fracture stress σ_f are about equivalent. In region C, the yield stress rises as the test temperature decreases, whereas the fracture

stress falls. The falling fracture stress is apparently associated with the increasing number of stable microcracks which are formed at the yield stress, and the fracture would appear to occur by the linking of these microcracks. Although the fracture appearance is 100% cleavage on a gross scale, microscopic observations have shown that there is considerable local plastic flow associated with the linking up of the microcracks.

Region B is defined by the limits T_m and T_t. Below T_t another mechanism of brittle failure is observed. The microstrain elastic limit is not observed, and the first indication of deformation is mechanical twinning which leads immediately to cleavage failure. Large local deformations are associated with these twins, and the details of the fracture mechanism have not been observed. It is significant to note, however, that stable microcracks have not been found below the microcrack transition, T_m. It appears, therefore, that the transition computed by Cottrell applies to T_m and not to T_d because the two conditions required in the computation are satisfied: (1) $\sigma_f \approx \sigma_y$ and (2) stable microcracks are observed above T_m and are not observed below. These conditions are only met in region B and are not satisfied in region C. In addition, it is unlikely that the fracture stress in region C has the assumed grain size dependence. The Cottrell theory appears to fit the microcrack transition temperature T_m quite well, and Eqs. 5 and 8 provide a convenient way of summarizing the influence of steel variable on the brittle-fracture problem.

The over-all picture is still probably incomplete. The role of mechanical twinning in region A is not understood and the mechanism of failure may be further complicated in materials with more complex microstructures. The energy factor p is also troublesome since it is now obtained from empirical correlations with the data, and the grain size and temperature dependence of this factor have not been explored. The cleavage facets also show peculiar "river" patterns that have been interpreted as showing the location of intersecting dislocations. In addition the role of the localized plastic deformation in linking up the cleavage facets is not well understood. Nevertheless, the dislocation picture of brittle failure has been very helpful in providing a theoretical framework for the mechanisms at play and it is expected that these concepts will continue to develop with more theoretical and experimental work.

ACKNOWLEDGMENTS

Figs. 1 through 4 are taken from the unpublished doctoral thesis work of George T. Hahn, and the experiment involving the search for microcracks in flat tensile bars is taken from the unpublished thesis work of William F. Flanagan. The author is very grateful for the opportunity to use these data. The author would also like to acknowledge many fruitful discussions with Morris Cohen and Walter Owen. A discussion with F. deKazinczy on the size of microcracks involved in the Cottrell theory is also gratefully acknowledged.

The new experimental work described here is sponsored by the Committee on Ship Steel of the National Academy of Sciences and their assistance is gratefully acknowledged.

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PHYSICAL METALLURGY AND MECHANICAL PROPERTIES OF MATERIALS: DUCTILITY AND THE STRENGTH OF METALLIC STRUCTURES

By. J. M. Frankland[1]

FOREWORD

The Engineering Mechanics Divisions Committee on Mechanical Properties of Materials conceived of the Symposium on Physical Metallurgy and Mechanical Properties of Materials as a means of summarizing, for the civil engineering profession, the current state of knowledge and some of the most recent developments in the understanding of materials behavior. The subjects of ductility, creep, brittle fracture, and fatigue have become of increasing concern in engineering applications, to the point where phenomenological descriptions of these aspects of materials behavior are becoming less than adequate for our needs.

The science of materials, stemming from physical chemistry, solid state physics, physical metallurgy, and mechanics of solids, gives us qualitative descriptions of the fundamental processes involved in the flow and fracture of solids. These fundamental concepts were presented by distinguished authorities in the 1956 Symposium on Physics of Engineering Materials organized by J. L. Waling and held at Pittsburgh, Pa. Frederick Seitz discussed imperfections in crystals, Thornton Reed, Jr., discussed dislocations, Clarence Zener discussed internal friction in metals, E. P. Blizzard and A. M. Weinberg discussed materials for radiation shielding, and G. J. Dienes discussed the effects of radiation on materials properties. The implications of these subjects in civil engineering practice were summarized by Glenn Murphy. Much of the material presented has been published in the literature of their own fields

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Note.—Discussion open until May 1, 1961. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. EM 6, December, 1960.

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and may be found in their published works. The applications of such fundamental concepts have brought about entirely new materials as well as improvement in traditional materials and indeed have created entirely new industries.

Theoretical relationships of the quantitative kind desired for engineering design purposes are, in many cases, still in the state of development. The development of such fundamental theory as exemplified by the study of brittle fracture by B. L. Averbach ("Brittle Fracture," Proceedings Paper, 2686) and the relaxation theory of creep of metals by F. H. Ree, T. Ree, and H. Eyring ("Relaxation Theory of Creep of Metals," Proceedings Paper 2333, Journal of the Engineering Mechanics Division, January, 1960, p. 41) and the importance of these concepts to engineering practice as indicated in the discussions of ductility by J. M. Frankland ("Ductility and the Strength of Metallic Structures," Proc. Paper 2687), of fatigue in structural materials by H. J. Grover ("Fatigue of Structural Materials," Proc. Paper 2688) and in the review of applications in civil engineering by Glenn Murphy ("Metallurgical Advances and Civil Engineering," Proc. Paper 2689) are considered by the committee to be invaluable to the profession of civil engineering.

In the present "Age of Materials," the problems of providing the materials suitable for the anticipated service conditions of many engineering applications require that all engineers be familiar with the latest and best of information pertaining to materials behavior. The practicing engineer must be cognizant of the rapidly growing field of materials science, which is so significant that, in some cases, it is causing large scale revisions of engineering college curricula.

The Committee on Mechanical Properties of Materials, through the 1956 Symposium on Physics of Engineering Materials, the 1958 Symposium on Physical Metallurgy and Mechanical Properties of Materials, the 1959 Symposium on the Physico-Chemical Nature of Soils, the 1960 Symposium on Nondestructive Testing of Materials, and the Symposium on Teaching of Materials in Civil Engineering Curricula being planned for 1961, have attempted to incorporate within the literature of the American Society of Civil Engineers a significant amount of the new knowledge in the field of materials behavior. Many of the participants in these symposia have been from professions other than civil engineering. A sincere expression of appreciation is hereby tendered to all who have contributed to this effort.

Joseph F. Throop, Chairman, E.M.D.
Committee on Mechanical Properties
of Materials, ASCE

SYNOPSIS

The need for the designer to consider ductility as an important factor in his design and not to merely assume it as being implicit in his design is emphasized in this paper. To illustrate this need the history of design prior to the discovery of mild steels is mentioned together with the early arguments of elastic versus plastic design.

The meaning of ductility and the factors that may affect its testing are enumerated. Four types of structural action in which some ductility of the mater-

ial is needed to carry the load are presented. Careful adherance to these factors is needed in order that a substantial reserve of ductility is maintained in the design of steel structures.

Ductility is a basic factor of materials performance to be taken into account in a philosophy of design. During the past seventy-five years (since 1885), design methods in mild steel have become sufficiently standardized that ductility is rarely considered explicitly by the designer. Nevertheless reliance upon ductility is implicit in much of the accepted practice in steel design.

To illustrate this point, one may take a look at the practices of a hundred years ago, before mild steel was available. The metals of primary construction then were cast and wrought iron. Cast iron is of limited ductility, while wrought iron is highly ductile. It is interesting to compare the differences in the application of these contrasting materials. It was recognized that cast iron was weak in tension, somewhat more satisfactory in bending, but most effective for compression members. Wide use was made of cast-iron columns. In Britain in particular, many small bridges were built with cast-iron arch ribs. Trusses frequently used cast-iron compression members with wrought-iron tie rods. The only available rolled shapes were small, so that girders of wrought iron were highly fabricated and thus expensive. Fairbairn's tests in the 1840's showed that wrought-iron beams were lighter than cast-iron beams of the same strength, but the cost was about equal. On the other hand, it was then much easier to produce interchangeable parts from castings than from riveted assemblies. This was a major factor in the decision to build the Crystal Palace with a cast-iron frame and a glass enclosure. This building was erected in Hyde Park, London, England, only nine months after initiation of working drawings, then disassembled and reerected on Sydenham Hill on the other side of London. This adaptability to prefabrication was a major factor in preferring cast-iron - in fact, for a time there was a brisk export trade in prefabricated structures of all kinds. Another factor, no doubt, was the adaptability of cast-iron members to ornamental detail of which the Victorians were so fond.

In spite of the practical advantages of cast iron, it was recognized that its indiscriminate use was wasteful of material, particularly in view of the highly variable strength properties of irons from different sources. The emergence of mild steel about 80 yr ago (about 1880) made it possible to develop heavier rolled sections at low cost and this material rapidly supplanted cast iron in most structural use.

This example shows that a low-ductility material such as cast iron may be used with advantage in compression members, but that other uses call for an extravagant use of material. The abrupt decline in the use of cast iron in the face of competition from a more versatile material (steel) indicates that the disadvantages of cast iron were quite obvious.

Two conflicting points of view regarding the design of structures prevailed during this period. The British engineers in general preferred to work on the basis of tests to failure of structural elements. The French, dominated by the theoretical elasticians of the period, preferred to design to the requirement that under actual working loads the structure remained entirely elastic. The factor of safety had an essentially different meaning to each school. The weight of academic authority was on the side of the French who condemned the British

as being mere empiricists. The French point of view eventually won out in civil and mechanical engineering, but not without making some concessions to the other side. These concessions are often lost to sight, because they are buried deep in practices that have the sanctions of many years of experience. There is (in 1960) a tendency, again, in several engineering fields, to use the conditions at failure as a criterion for design, and thus to introduce considerations of plastic deformations to supplement elastic analysis. In a sense, this is a return toward the point of view of the engineers of nineteenth century Britain.

What is meant by ductility? The answer to this from most engineers would be the elongation and reduction of area measured in a tensile test, with the usual emphasis being on the elongation. This, however, is not adequate for all structures. Consider a square bar of steel bent around a mandrel of small diameter. The elongations suffered by the extreme fibers before failure are much greater than the elongation observed in the tensile test. They can approach the local elongation observed in the neck of the tensile specimen, a quantity measured by the reduction of area. This may be very large, say 80% or more. The reason for this is that necking is largely a peculiarity of the tensile test, and the elongation as usually measured may be greatly exceeded if necking does not occur, as in the bend test. A more important matter is that ductility is not merely a property of the material, but depends on the system of stresses that is imposed. Tremendous increases in reduction of area are found, for example, when tensile tests are conducted under high hydrostatic pressure. On the other hand, sharply notched specimens of high-strength steel break with very little ductility. If it were practicable to do such an experiment, failure of a metal under uniform hydrostatic tension would occur without plastic flow, regardless of the ductility shown in a tensile test. A not uncommon case is one in which the stresses are biaxial and tensile, and one is half of the other. Such occurs in cylindrical pressure vessels where the longitudinal tension is half the hoop tension. Fractures in such a vessel occur with a considerable drop in ductility—a representative figure being half the maximum local elongation observed in simple tension. In about 1930, it was the practice to deliver industrial gases at 3,000 psi pressure in mild steel bottles. A number of fractures occurred at low temperatures when such bottles were dropped. This was due to the reduced ductility under the system of stresses in the bottle plus the enhancing effect of low temperature, about which will be reported in detail elsewhere in this program. As a result, these bottles today are charged only to about 2,000 psi, the bottles themselves being unchanged.

In short, then, the ductile or brittle behavior of a material is greatly affected by the stress system to which it is subjected and the simple tensile test gives only one indication of the behavior of the material. Other tests are needed to round out the picture. Tests with sharp circumferential notches in round tensile specimens have been found useful in descriminating among steels of high tensile strength. Edge-notched plates of mild steel at low temperatures have given us valuable information on brittle fracture. The Charpy notch impact test, which is a notch test rather than an impact test, has also been useful in distinguishing between mild steels subject to brittle fracture.

It is not suggested that such a testing program is necessary to substantiate the general use of mild steel in structures. On the contrary, at normal temperatures most mild steel exhibits an ample reserve of ductility to provide against the adverse effects of multiaxial stresses. It is only the unusual combinations of ambient conditions, structure, and material that demand scrutiny.

Low ductility may be present only in certain directions of testing of a single block of metal. These directional effects are most noticeable in heavy sections such as billets or slabs. Usually the ductility of a tensile specimen cut parallel to a minor dimension will be less than the ductility parallel to a major dimension. The ductility measured along the thickness direction of a slab can be surprisingly low. This behavior seems to be a general characteristic of forged metals. Difficulties of this kind have been encountered by the aircraft industry when using fittings machined from forged blocks of aluminum alloy. Some of the failures of large turbine rotors may be associated with similar conditions. One should therefore be wary of the use of heavy sections unless one is sure that the ductility and strength are adequate in all directions along which there are appreciable stresses. Castings can be used to minimise the directional effect, but they are subject to other drawbacks, such as blow holes and inhomogeneities, which often make them as unacceptable substitute.

Since many materials of low ductility show marked notch sensitivity in fatigue, it has frequently been suggested that good ductility is needed for strength under repeated loading. Tests have not borne this out in general. On the other hand, materials with marked directional differences in ductility have at times shown unusually short lives in fatigue loading. Some tests by the writer suggest that this is due to exceptionally rapid growth of fatigue cracks. The initial crack can form at an early stage in the life of a specimen. After this, anything that accelerates crack growth will lead to a shorter fatigue life. Low ductility in such cases appears to be not a cause, but a concomitant result of some defective structure in the material.

It is generally recognized today that joints in a fabricated structure are points of stress concentration that are rendered harmless by small amounts of plastic deformation, highly localized. That this was not always so realized is indicated by an editorial in "Engineering" (around 1910), in which the editor describes a lecture by M. Coker on photoelastic determinations of the large stress concentrations around a hole in plate under tension. The editor comments that this is all very well for theoreticians, but no practicing engineer could believe in such things, otherwise the Firth of Forth bridge would have fallen down long ago!

One may make a rough conservative estimate of the ductility needed to eliminate stress concentrations in the following way. When sufficient load is applied to bring the entire cross section of the member to the yield, one may consider that the stress concentrations are virtually eliminated. If ϵ_Y is the elastic strain at yield and n is a strain concentration factor, the maximum strain in the region at the stress concentration will be of the order of $n\,\epsilon_Y$. To compare this with the ductility shown in uniaxial tension, we must multiply by another factor k which reflects the reduction in ductility possible under multiaxial stress. A representative value for k can be taken as two. Thus, to withstand a local strain concentration of n, we need a uniaxial ductility of approximately $2\,n\,\epsilon_Y$. Tests show the possibility that the strain concentration may be somewhat higher than the stress concentration. As a value indicative of the conditions in a riveted or bolted joint, we might take $n = 5$. For such a case we need a ductility in the tensile test of $10\,\epsilon_Y$. For mild steel this amounts to about 1%. Even for material that has had to submit to some exhaustion of ductility in fabrication processes, it is clear that mild steel under normal conditions has plenty of reserve to meet this requirement. Severely deformed punched holes may be an exception to this remark.

It is important to note that the ductility required is proportional to the maximum elastic strains in the material. For materials other than mild steel, the required ductility may be higher. As an example, for the strong 7075-T6 aluminum alloy, the same conservative estimate yields about 6%, and a similar amount is needed for the steels of very high strength now being used in aircraft. It will be seen that the larger the elastic range in terms of strain, the more ductility is required.

Most acceptable structural materials in present use will then have an ample reserve of ductility so as to be unimpaired in strength by stress concentrations at the connections. Only in materials of low ductility will such stress concentrations affect the strength. Such occasions can arise with large forgings where the ductility in some directions may be low. Plates loaded in tension through the thickness have sometimes given rise to unanticipated failures of this kind.

We have spoken earlier of a trend in structural practice to design in terms of the load at failure of the structure. Design on this basis must take account of plastic behavior. On this basis redundant structures are found to possess reserves of strength not shown by statically determinate structures. This point of view dates back to the beginning of the 20th century, but has received explicit formulation by J. A. Van den Broek[2] and more recently by other engineers, in particular J. F. Baker.[3] To a large extent this attitude represents the philosophy of the aircraft designer who deliberately seeks to predict the actual behavior of this structure at failure. Because of the economics of aircraft construction, the aeronautical engineer has the advantage of frequent tests to destruction of his assemblies, so that he can be fully aware of the modes of failure of his designs. The civil engineer on the other hand is justifiably reluctant to depart from well-established practice until a sufficient volume of test results has been accumulated to give him confidence. Many such tests, however, are under way and enough data have been obtained to show that the point of view of limit design will have important effects on the design philosophy of civil engineers.

Limit design often leads to different factors of safety than would be predicted from conventional elastic design. A rolled shape tested as a simply-supported beam will collapse when the flanges have become fully plastic at the point of maximum bending moment. This condition is reached for normal shapes at about 115% of the bending moment which produces first yielding. It is assumed, of course, that the beam is adequately supported against premature failure by lateral instability. As the moment distribution is unaffected by yielding in a simple beam, elastic design and limit design in this case lead to the same proportions and the same factor of safety. The situation is quite different, however, in a fixed-end beam under uniform load. Here the greatest bending moments under elastic conditions are at the built-in ends and yielding begins first at these points. When the ends have reached their maximum plastic moment, the rest of the beam is still elastic and more load can be applied without large increase in deflection, the end moments remaining constant and the center moment increasing. Collapse finally occurs when the bending moment at midspan reaches its full plastic value. Elastic design ignores the action beyond the point at which the end moments develop plastic behavior. Limit design, on the other hand, considers the redistribution of bending moment that

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[2] "Theory of Limit Design," by J. A. Van den Broek, Wiley, New York, 1948.

[3] "The Steel Skeleton," Vol. II, by J. F. Baker, M. R. Horne, and J. Heyman, Cambridge Univ. Press, New York, 1956.

takes place due to plastic action at the ends. In this case, the actual failure load is one-third higher than would be predicted from elastic design. The advantages of limit design and its close relation to the designer's problem of choosing the initial proportions of his structure have been discussed recently (1957) by D. C. Drucker.[4]

The ductility required in this approach to design is that which will develop the full plastic moment of the section. For mild steel, this seems to be slightly more than the plastic elongation at the yield point before the stress-strain curve starts to rise again—that is, 1 1/2% to 2%. Usually the factor of safety will be so chosen that no plastic action of the kind considered will occur under maximum working loads, but sufficient ductility must be present to permit the development of the postulated kind of behavior under overloading.

The importance of secondary stresses to the strength of structures has been a matter of considerable argument. J. I. Parcel and E. B. Murer have reviewed this problem.[5] They concluded that in most cases secondary stresses are eliminated by small amounts of plastic deformation. Like the end moments of the fixed-end beam under distributed load, they are not essential to static equilibrium, so local relief by plastic deformation does not affect the strength. This process is quite similar to the plastic deformation envisaged in limit design, so the ductility requirement is substantially the same, let us say another 2% for mild steel.

Of course, to take advantage of the plastic readjustments in limit design and in the relief of secondary stresses, care must be taken in the design of connections and in proportioning members to avoid local crippling and lateral instability. It appears that this can be assured without appreciable departures from accepted practice.

High residual stresses are known to be present in rolled sections and in welded assemblies. These can give rise to plastic yielding at low values of the applied loads. The process of plastic readjustment in this case is similar to what occurs at stress concentrations. Just as in the former case, the plastic strain required is proportional to the elastic range of strain. A ductility allowance equal to that demanded by stress concentrations would seem to be ample.

We have discussed four types of structural action in which some ductility of the material is needed to carry the load. These are: (1) plastic relief around stress concentrations, (2) redistribution of internal lodas in redundant structures, (3) relief of secondary stresses, and (4) plastic flow under combinations of load stresses and high residual stress. The first and last items call for a plastic deformation proportional to the elastic strain range. For mild steel it is suggested that each type of behavior calls for approximately 1% of plastic strain. For other materials this requirement could be greater. The second and third classes of behavior call for a ductility less dependent on the particular material. For mild steel an allowance of 2% each seems sufficient. Thus, under the worst combination of circumstances, a mild steel structure under load might have to develop as much ductility as that represented by 6% elongation in the tensile test.

It should not be forgotten that fabrication and erection make demands on the ductility of the material, so an additional allowance should be made for this.

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[4] "Plastic Design Methods - Advantages and Limitations," by D.C. Drucker, Transactions, Soc. Naval Architects and Marine Engrs., Vol. 65, 1957, p. 172.

[5] "Effect of Secondary Stresses upon Ultimate Strength," by J. I. Parcel and E. B. Murer, Transactions, ASCE, Vol. 101, 1936, 289.

Perhaps we might estimate that mild steel needs overall at least 10% ductility in the tensile test to perform adequately in the usual design. Fabrication practices such as shearing and punching without subsequent machining may call for a greater amount of ductility.

The preceding estimate of the ductility needed is conservative and is based upon present practices in ordinary steel structures. Where it is possible to use special precautions, these requirements can be reduced. Careful design of details to minimize stress concentrations and secondary stresses and careful handling in fabrication and erection can make materials of lower ductility acceptable, and this has been done in the aircraft industry. However, this is a prickly path, and the designer will breathe more freely if he is dealing with a material having a substantial reserve of ductility.

We have hitherto discussed plastic behavior as a helpful effect that makes it possible for stress peaks to be smoothed out, thus permitting the structure to function in simple and direct means of carrying load. There is another and equally important role that it plays, and that is, to set a limit on the useful response of a structure under load. Thus, a statically determinate structure cannot carry useful loading beyond the point at which one of its members begins to yield over the full cross section. This is analogous to the simple beam discussed earlier. Columns in mild steel cannot exceed an average stress equal to the yield point at most before failure. At intermediate lengths, columns will fail at even lower loads. This may be understood only if we take plastic effects into account.

It therefore seems abundantly clear that we cannot understand the real behavior of a metallic structure, either at working loads or at failure, without taking into consideration the capacity of the material for plastic deformation. Hence no realistic design philosophy can afford to ignore the particular behavior of a material as expressed by its ductility.

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PHYSICAL METALLURGY AND MECHANICAL PROPERTIES OF MATERIALS: FATIGUE OF STRUCTURAL MATERIALS

By Horace J. Grover[1]

FOREWORD

The Engineering Mechanics Divisions Committee on Mechanical Properties of Materials conceived of the Symposium on Physical Metallurgy and Mechanical Properties of Materials as a means of summarizing, for the civil engineering profession, the current state of knowledge and some of the most recent developments in the understanding of materials behavior. The subjects of ductility, creep, brittle fracture, and fatigue have become of increasing concern in engineering applications, to the point where phenomenological descriptions of these aspects of materials behavior are becoming less than adequate for our needs.

The science of materials, stemming from physical chemistry, solid state physics, physical metallurgy, and mechanics of solids, gives us qualitative descriptions of the fundamental processes involved in the flow and fracture of solids. These fundamental concepts were presented by distinguished authorities in the 1956 Symposium on Physics of Engineering Materials organized by J. L. Waling and held at Pittsburgh, Pa. Frederick Seitz discussed imperfections in crystals, Thornton Reed, Jr., discussed dislocations, Clarence Zener discussed internal friction in metals, E. P. Blizzard and A. M. Weinberg discussed materials for radiation shielding, and G. J. Dienes discussed the effects of radiation on materials properties. The implications of these subjects in civil engineering practice were summarized by Glenn Murphy. Much of the material presented has been published in the literature of their own fields and may be found in their published works. The applications of such funda-

Note.—Discussion open until May 1, 1961. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. EM 6, December, 1960.

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mental concepts have brought about entirely new materials as well as improvement in traditional materials and indeed have created entirely new industries.

Theoretical relationships of the quantitative kind desired for engineering design purposes are, in many cases, still in the state of development. The development of such fundamental theory as exemplified by the study of brittle fracture by B. L. Averbach ("Brittle Fracture," Proceedings Paper, 2686) and the relaxation theory of creep of metals by F. H. Ree, T. Ree, and H. Eyring ("Relaxation Theory of Creep of Metals," Proceedings Paper 2333, Journal of the Engineering Mechanics Division, January, 1960, p. 41) and the importance of these concepts to engineering practice as indicated in the discussions of ductility by J. M. Frankland ("Ductility and the Strength of Metallic Structures," Proc. Paper 2687), of fatigue in structural materials by H. J. Grover ("Fatigue of Structural Materials," Proc. Paper 2688) and in the review of applications in civil engineering by Glenn Murphy ("Metallurgical Advances and Civil Engineering," Proc. Paper 2689) are considered by the committee to be invaluable to the profession of civil engineering.

In the present "Age of Materials," the problems of providing the materials suitable for the anticipated service conditions of many engineering applications require that all engineers be familiar with the latest and best of information pertaining to materials behavior. The practicing engineer must be cognizant of the rapidly growing field of materials science, which is so significant that, in some cases, it is causing large scale revisions of engineering college curricula.

The Committee on Mechanical Properties of Materials, through the 1956 Symposium on Physics of Engineering Materials, the 1958 Symposium on Physical Metallurgy and Mechanical Properties of Materials, the 1959 Symposium on the Physico-Chemical Nature of Soils, the 1960 Symposium on Nondestructive Testing of Materials, and the Symposium on Teaching of Materials in Civil Engineering Curricula being planned for 1961, have attempted to incorporate within the literature of the American Society of Civil Engineers a significant amount of the new knowledge in the field of materials behavior. Many of the participants in these symposia have been from professions other than civil engineering. A sincere expression of appreciation is hereby tendered to all who have contributed to this effort.

Joseph F. Throop, Chairman, E.M.D.
Committee on Mechanical Properties
of Materials, ASCE

SYNOPSIS

The current state of knowledge about the fatigue properties of structural materials is reviewed. The progress being made, in the case of metallic alloys, toward relating fatigue damage to the behavior of dislocations in the crystalline lattice is enumerated. The need for more information on the behavior of nometallic materials under repeated stressing is emphasized.

Methods of studying fatigue, the nature of fatigue, and the extent of laboratory fatigue tests are indicated. It is concluded that present knowledge indi-

cated strongly that fatigue involves very local weaknesses and that design to prevent fatigue will require more detailed consideration than previously has been necessary for design for static strength.

INTRODUCTION

Over a hundred years ago, it was observed that structural parts of mild steel sometimes fractured after many repetitions of a stress considerably less than the steady stress that they could support. This behavior was termed "fatigue."

Since then, fatigue failures have been reported in many types of machine parts and structural components. Failures attributed to fatigue from repeated stresses have involved large economic losses and, in a few instances, catastropic accidents. Susceptibility to fatigue under repeated stresses has been demonstrated for many structural materials, including metals and metal alloys, woods and plywoods, mortars and concretes, and laminated plastics.

Advancing technology provides a number of potential new sources for fatigue. There are increasing sources of vibration and stress repetition. At the same time, there are trends for closer design to minimize weight, to decrease material costs, and to permit more economical fabrication and assembly. Moreover, there appear new materials and new uses for old materials. Consequently, it becomes increasingly important for the engineer to know as much as possible about design in order to prevent fatigue failure of structural components that must withstand repeated loading. The objective of this paper is to review the current state of knowledge about the fatigue properties of structural materials.

In accordance both with the interests of this Symposium and with the bulk of available information, the review will emphasize knowledge about metals and alloys. However, it is proposed to note (partly by way of contrast) some of the relatively limited information concerning fatigue behavior of nonmetallic materials.

For brevity, the discussion will concern fatigue at room temperature and in the absence of corrosive environment or surface fretting. Fretting, chemical corrosion, elevated temperatures, and thermal stresses present added complexities of increasing engineering importance. However, even a brief review of the behavior of materials under repeated loading in absence of these complexities contains enough challenging questions for present consideration.

METHODS OF STUDYING FATIGUE

The phenomenon of fatigue of metals under repeated loads was an engineering discovery motivated by practical problems. Early reports were concerned with such items as mine-hoist chains (1829), bridges (1842), and axles of railway vehicles (1848-1852). Investigations of fatigue of nonmetallic materials were reported somewhat later, but were also related to practical problems (for example, studies of mortars in 1898).

In the ensuing years, further studies have often been concerned with engineering needs and have been directed toward accumulating data for engineering design of particular machine parts and structural components. Some information has been deduced from field experience and some has been obtained by laboratory tests of parts and components under simulated-service loading.

Such information has been most useful in many instances, but is not often easily generalized for wide application.

Particularly in the past 25 yr (since about 1935), a considerable body of data on various materials has accumulated from systematic laboratory experiments on test pieces subjected to controlled cyclic loading. Many of the results are available in published literature.[2,3,4,5] These results afford a great deal of information on the comparative merits of different materials and of different methods of fabricating a specific material. The data from such laboratory tests also furnish empirical rules formulated for design.

A number of investigators have sought to understand the basic mechanism whereby materials fracture under repetitions of loads which, if static, would not cause failure. Accounts of such studies are available.[6,7,8] It must be admitted that the mechanism is not yet wholly understood, although (as will be suggested) progress has been made.

THE NATURE OF FATIGUE

From the many observations recorded over the past century, it seems that the process of fatigue failure can be usefully described in three phases.

1. Under successive cycles of stressing, cumulative damage occurs in some small region until a very tiny crack exists there.
2. Under additional cycles, this crack propagates until it becomes visible without magnification, and then it grows until it has weakened the section.
3. After a few additional cycles, the specimen fractures.

Much effort has been expended towards description and understanding of the first step.

E. Orowan[9] has pointed out that the inception of a fatigue crack may result from (1) the two processes of strain hardening and some sort of damage, and (2) the presence in any material of localized weak spots. For metals, there is a start toward understanding strain hardening, somewhat less certainty about the damage process, and adequate possibilities for weak spots.

Particular efforts have been devoted to searching for evidence of damage prior to cracking in relatively pure (and ductile) metals. Considerable evidence indicates the importance of localized slip that is different under cyclic loading than under steady loading. Present belief (1960) is that this may be understandable in terms of dislocations in the crystal lattice. It is not clear that localized slip will explain repeated stress damage in very brittle mate-

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2 "Fatigue of Metals," by R. Cazaud, Chapman and Hall, London, 1953.

3 "The Fatigue of Metals and Structures," by H. J. Grover, S. A. Gordon, and L. R. Jackson, U. S. Government Printing Office, 1954.

4 "Fatigue of Concrete," by G. M. Nordby, Annual Meeting of American Concrete Institute, 1958.

5 "References on Fatigue," American Society for Testing Materials, Committee E-9, Subcommittee III (annually, 1950-date.)

6 "Deformation and Fracture of Mild Steel Under Cyclic Stresses in Relation to Crystalline Structure," by H. J. Gough and W. A. Wood, Institution of Mechanical Engineers, Journal and Proceedings, Vol. 141, pp. 175-185, 1939.

7 "Conference on Fatigue and Fracture of Metals," Massachusetts Institute of Technology, Chapman and Hall, London, 1951.

8 "International Conference on Fatigue of Metals," Institution of Mechanical Engineers, London, and American Society of Mechanical Engineers, New York City, 1956.

9 "Theory of the Fatigue of Metals," by E. Orowan, Proceedings of London Royal Society, Vol. 171A, p. 7a, 1939.

rials, and A. M. Freudenthal[10] has proposed a probability sequence of breaking weak bonds for such materials. For fibrous materials with long organic chain structures, few speculations have been reported.

It is quite possible that the propagation of a crack under cyclic loading may follow somewhat different stress relations than the inception of the crack. In the case of metals, present studies of crack propagation under static and impact loadings may eventually be helpful toward understanding the second step in development of fatigue failure.

Since there is incomplete knowledge of the mechanism of fatigue, present engineering depends mainly upon empirical principles involving interpolation between, and sometimes speculative extrapolation from, observed test data. The one precept from current indications about the mechanism that appears of great importance is that fatigue seeks out very localized weak spots in a test piece or in a structure. Consequently, consideration of fatigue requires attention to small details that are less important under most steady stress conditions.

LABORATORY FATIGUE TESTS

Most laboratory studies have involved subjecting a number of test pieces to different magnitudes of cyclic stress (or of cyclic stress superimposed upon a selected steady stress) and recording the different lifetimes to failure. Since such tests usually show considerable scatter of results, current practice emphasizes statistical planning and interpretation of results.[11] Types of specimens, types of loading, and environmental conditions have been dictated partly by convenience of testing and partly by anticipated service requirements. Accordingly, the nature and extent of information on different materials differ widely.

Most structural materials have been tested under some form of cyclically reversed bending. Results are usually plotted on a stress versus number-of-cycles-to-failure (S-N) graph. Fig. 1 shows illustrative results for several materials. Fig. 2 shows the same data with values of ratio of stress amplitude to static failure stress as ordinates. In each figure, the term "stress amplitude" refers to one-half the range from minimum stress to maximum stress. This and other terms used in this paper follow American Society for Testing Materials (ASTM) nomenclature for fatigue.

While the values shown in Fig. 1 and 2 should be considered illustrative rather than absolute, they provide some interesting observations.

1. Each of the several quite different materials appears to fail after many repetitions of a stress cycle in which the maximum stress is significantly lower than the static strength of the material.
2. For the several materials shown, fatigue strengths for 10-million load reversals range from about 30% to 55% of static strength.
3. There are observable differences in the shapes of the S-N curves for the different materials.

An item that does not appear on these figures is the fact that for most materials in absence of special conditions the fatigue strength is dependent on the number

10 "The Characteristics of Fatigue of Metals," by A. M. Freudenthal and T. J. Dolan, Fourth Progress Report to ONR, Contract N6 ori-71, February, 1948.

11 "Guide for Fatigue Testing and Statistical Analysis of Fatigue Data," American Society for Testing Materials, Committee E-9, Task Force on Statistics, 1957.

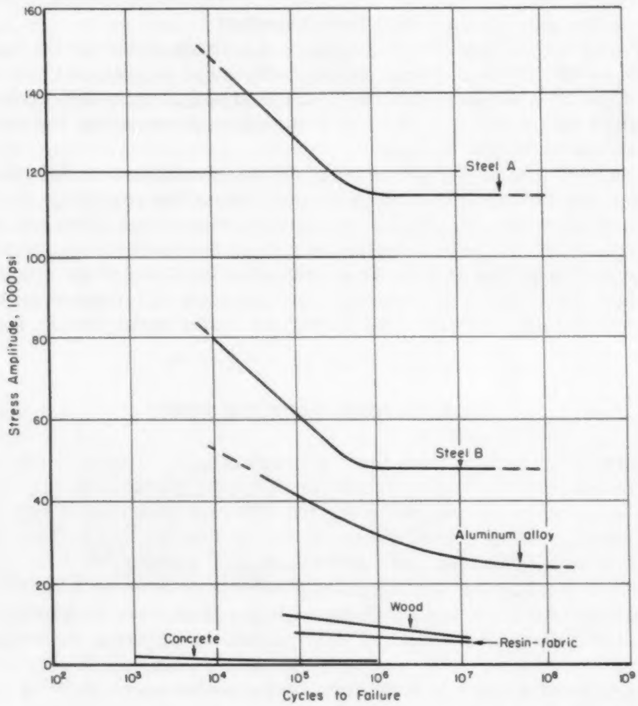

FIG. 1.—REVERSED-BENDING FATIGUE-TEST RESULTS FOR DIFFERENT MATERIALS

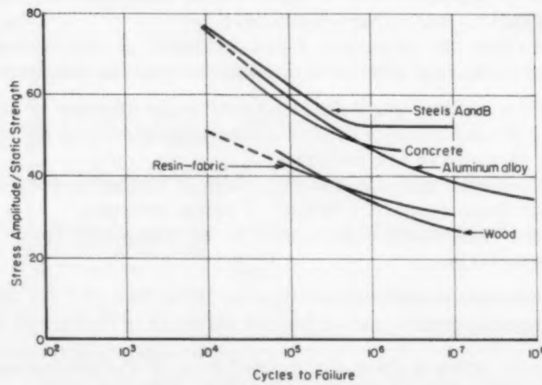

FIG. 2.—REVERSED-BENDING FATIGUE STRENGTHS AS PERCENTAGES OF STATIC STRENGTHS OF DIFFERENT MATERIALS

of cycles rather than on time; that is, it is relatively insensitive to speed of cycling over an appreciable range of frequencies. Special conditions might include the following: for metals, elevated-temperature or corrosive environment; for woods, atmospheric humidity; for concretes, humidity and length of curing.

Most structural components involve other considerations than the behavior of a smooth beam under fully reversed flexure. Loadings of interest involve such concerns as (1) a steady load upon which is superimposed an alternating load, (2) torsion or torsion combined with tension, and (3) often randomly varying loads. Critical configurations almost always include a stress raiser, such as a notch, a rivet, or another type of fastener. Because different materials have been most often studied for the load conditions and stress raisers of most common importance in their use, it is difficult to discuss available information on such matters for structural materials in general. Accordingly, subsequent illustrations are given for specific materials.

FATIGUE BEHAVIOR OF CARBON STEELS AND OF LOW-ALLOY STEELS

The fatigue behavior of mild steels has been investigated extensively. Most widely used steels have been tested in rotating bending with particular attention to the fatigue limit (the stress at which the S-N curve is essentially horizontal; note Figs. 1 and 2). Through a wide range of composition and heat treatment, there seems to be a general correlation between the fatigue limit and the ultimate tensile strength. Up to moderate values of tensile strength (perhaps 200,000 psi), values of fatigue limit range around 50% of the tensile strength; above this, they may be lower. The considerable scatter in any plot of data for this relation has been attributed to variations in ductility, metallurgical notches, and internal stresses.

Axial-load fatigue tests of steels have been run so as to provide a series of S-N curves for different parametric values of mean stress. From such a series there may be derived a Soderberg (sometimes called "Goodman") diagram, indicating the effect of mean stress upon the stress amplitude that can be withstood for a specified lifetime. Fig. 3 illustrates such a diagram. It may be noted that, particularly for long lifetimes, the mean stress has relatively little effect until the maximum stress in a cycle exceeds the yield point. Relatively little information is available for conditions in which the mean stress is compression. Some evidence suggests that under such conditions the tolerable stress amplitude increases.

There has been considerable interest in fatigue behavior of steels under combined stresses. Several investigators have sought to find a method of predicting fatigue strength under any system of stresses when it has been determined for one type of stress. Most experimental work has been done under fully reversed bending and torsion, and analyzed in terms of familiar strength theories. Three such theories and corresponding predicted ratios of torsion-fatigue strength, σ_T, to bending-fatigue strength σ_b, are: (1) Principal-stress theory, ($\sigma_T/\sigma_b = 1.00$); (2) Manimum-shear-stress theory ($\sigma_T/\sigma_b = 0.50$); and (3) Shear-stress-invariant theory; $\sigma_T/\sigma_b = 0.58$. For several steels, experimental values of this ratio vary from 0.53 to 0.86 with an average value of 0.63. This apparent importance of shear stress seems pertinent to observations of subsurface inception of fatigue under rolling loads (bearings, wheels, rails, and so on). Studies of behavior of steels under biaxial tension have generally shown such large effects of anisotropy that only empirical interpolation

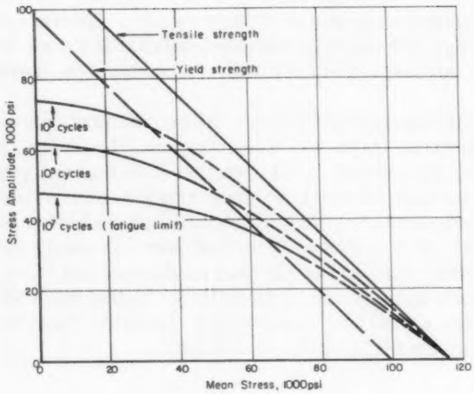

FIG. 3.—SODERBERG DIAGRAM FOR A STEEL

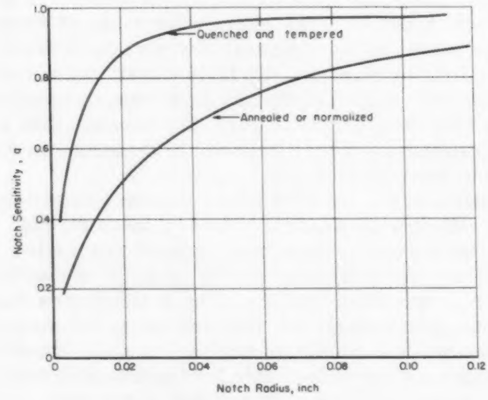

FIG. 4.—NOTCH-SENSITIVITY INDEX FOR STEELS IN ROTATING BENDING (AT FATIGUE LIMIT)

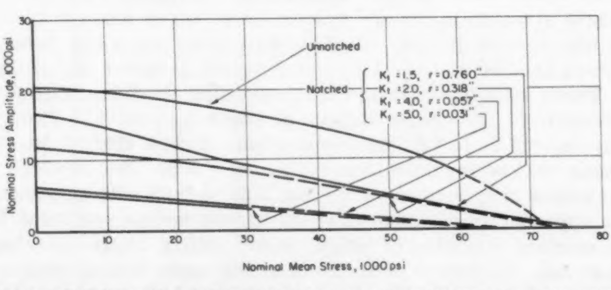

FIG. 5.—SODERBERG DIAGRAM FOR AN ALUMINUM ALLOY (SHEET SPECIMENS, EDGE NOTCHES, LIFETIME 10^7 CYCLES)

between measured values for various combinations have been successful. Few investigations of combined stresses under non-fully-reversed loading have been reported.

Many observations indicate that the fatigue strengths of steels are much more critically influenced by notches than are the static strengths. This effect has been studied by comparing S-N curves for unnotched specimens with those for notched specimens. For fully reversed loading, the "fatigue notch factor" is defined as

$$K_f = \frac{\text{Fatigue strength of unnotched specimen}}{\text{Fatigue strength of notched specimen at same number of cycles}}, \quad \ldots \quad (1)$$

in which the stress amplitude for the notched specimen is in terms of nominal stress (Mc/I, P/A, Tc/J) at the section of the notch. For a specific steel, K_f usually increases with the severity of the notch, as indicated by its theoretical stress-concentration factor, K_t. The quantity

$$q = \frac{K_f - 1}{K_t - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

is often used as a measure of the "fatigue notch sensitivity" of the steel. Fig. 4 shows some values of q varying with notch radius, which seems an important item. P. Kuhn and H. F. Hardrath[12] have suggested a different approach.[8] In engineering design, any source of stress concentration (a geometric notch or change in section, a metallurgical discontinuity such as an inclusion or the heat-affected zone of a spot weld, or a localized stress at a bolt or rivet) may produce the deleterious effect of a sharp notch.

It has been found experimentally that a specimen repeatedly stressed (short of failure) at some level may have, at another level of repeated stress, a lifetime different from that of a virgin specimen. Considerable effort has been spent toward finding a way of assessing the cumulative fatigue damage to a steel subjected to varying stresses of different magnitudes as so often encountered in service. At present, there is no satisfactory solution to this very practical problem. It appears that better approaches to the question of cumulative damage may become apparent when a little more insight into the basic mechism of fatigue has been obtained.

In summary, it appears that the fatigue properties of the widely used steels are relatively insensitive to composition, somewhat dependent on metallurgical structure for a particular hardness, and very approximately proportional to hardness (or ultimate tensile strength). (While results are not conclusive, it appears that a tempered martensitic structure is better in fatigue than one containing intermediate transformation products.) These trends may be offset, in a particular instance, by other factors such as surface condition (including the important item of possible decarburization or excess carburization), residual stress, inclusions, or any other stress-raising items. Such factors may vary from one heat to another, and the fatigue strength of a part may be criti-

[12] "An Engineering Method for Estimating Notch-Size Effect in Fatigue Tests on Steel," by P. Kuhn and H. F. Hardrath, NACA TN-2805, October, 1952.

cally influenced by factors in the heat from which it is made and by its fabrication or heat treatment or combination of the two.

FATIGUE BEHAVIOR OF ALUMINUM ALLOYS

Particularly in view of the demanding requirements in aircraft, aluminum alloys have been tested extensively in fatigue.

Most commonly used alloys have been tested in rotating bending to lifetimes of 5×10^8 cycles. It is usually stated that these alloys do not have a true fatigue limit (compare the curve for an aluminum alloy with the sharply flattened curves for steel in Fig. 1). Another characteristic is that the long-lifetime fatigue strength does not show the trend of increasing with increasing ultimate tensile strength that steels show. An average upper limit for the fatigue strengths of aluminum alloys is about 20,000 psi. For the higher (static) strength alloys, this corresponds to about 30% of the tensile strength. It has been suggested[8] that precipitation during the cyclic stressing may influence the relatively low fatigue strength of the high-strength aluminum alloys.

The wrought alloys, used widely in sheet form, have been investigated rather extensively in axial loading with varying combinations of mean and alternating stress. Fig. 5 shows a Soderberg diagram from data on sheet specimens of a wrought alloy. The curves are drawn for a lifetime of 10^7 cycles and include results for unnotched specimens and for specimens with edge notches of different severities. In general, the effect of mean stress is similar to that for steels. As for steels, notches reduce fatigue strength severely, and the fatigue notch factor appears to increase with K_t but to be also influenced by notch radius. There are, however, quantitative differences[13] concerning the variation of q with notch radius for steels and for aluminum alloys.

Information on the effect of combined stress states and information on cumulative damage in fatigue is not much more or much less complete for aluminum alloys than for steels. One factor important in some uses is the effect (a reduction in fatigue strength) of the soft cladding commonly used for corrosion protection of high-strength aluminum alloys. In general, sufficient data are available to take this into account in design evaluations.

FATIGUE BEHAVIOR OF OTHER METALS AND ALLOYS

There are considerable data on the fatigue properties of several other metals and alloys, including cast irons, magnesium and its alloys, cooper and copper-base alloys, nickel and nickel-base alloys, and some stainless steels and other heat-resisting alloys. There are some data on relatively new metals and alloys and, in fact, a few data on nearly every alloy of engineering interest. As already noted, the extent and nature of fatigue investigation of each has been mainly related to engineering interest in anticipated uses of that materials.

Most metals and alloys show trends within the range of those indicated for steels and aluminum alloys. As might be expected, however, details vary with the metallurgical behaviors of the alloy systems. For steels, the nature and distribution of the carbides which determine the static strength seem to determine also the fatigue strength. For many aluminum alloys, the precipitation

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13 "Effect of Geometric Size on Notch Fatigue," by P. Kuhn, International Union of Theoretical and Applied Mechanics, Colloquim on Fatigue, in Stockholm in 1955, Julius Springer, Berlin, 1956.

hardness obtained by suitable composition and heat treatment apparently influences fatigue strength less than static strength. Magnesium alloys, because of their hexagonal crystal structure, have less favorable cold-working characteristics than cubic metals such as aluminum, copper, and iron. In some instances, this may be reflected in notch-fatigue behavior. Some cast irons show low fatigue-notch sensitivity (and low fatigue strengths) that has been ascribed to the character and distribution of the graphite flakes which act as metallurgical notches. In general, the fatigue strength of a metal is sensitive to details of metallurgical structure and to inhomogeneities in the structure.

It is important to realize that practically controllable factors such as machining, surfacing, welding, imposition of residual stresses as well as composition and heat treatment are related to the fatigue strength of a metal or alloy in a manner peculiar to the metallurgical nature of that material. This means that interpolations and extrapolations of specific data as well as experimental evaluation of a new material or process should be made with reasonable understanding of the metallurgical factors likely to be involved.

FATIGUE BEHAVIOR OF NONMETALLIC MATERIALS

With the motivation of many practical problems, there have not only been extensive studies of the fatigue of metals but also several collections of pertinent data and published accounts of progress in such studies.[2,3,5,7] For most nonmetals, there is less available information and the existing data have been less extensively collected and correlated.

Recently (1958), there has been a careful review of investigations of the fatigue studies of concrete.[4] The fatigue strength, like other strength properties, of plain concrete is affected by many factors, such as the size and shape of aggregate, the compaction, the moisture content, and environmental conditions. There seems to be a definite fatigue behavior: a propensity to fail under repeated flexure at stresses of the order of 55% of the modulus of rupture. In some cases, the term "repeated flexure" in investigations of concrete does not imply fully reversed loading. In a few instances of fully reversed loading (including that used for illustration in Figs. 1 and 2), fatigue strengths lower than 55% of the ultimate have been reported.) For a range of speeds (70 cpm to 440 cpm), failure appears dependent on the number of cycles rather than on the time; at slower speeds, a "creep phenomenon" seems interrelated, while no data exist at higher speeds. There is some evidence of an effect of mean stress upon range of stress; little information on effects of combined stress; there are few data on possible notch effects. A considerable amount of work has been done on the complex study of reinforced concretes (both without and with pre-stressing); in reinforced concrete, failure of reinforcements and bond failure are important contingencies. More research is needed toward understanding the mechanism of fatigue in either plain or reinforced concrete; pending such understanding, further systematic experiments may provide useful engineering-design information that could bring considerable economic returns.

Another group of nonmetallic materials which have been investigated in fatigue includes woods and plywoods. Much of the work on these materials has been done at the United States Forest Products Laboratories (and is described in various reports from there). Here, too, factors (anisotropy with respect to grain direction, moisture content, etc.) which influence static strengths, significantly affect fatigue strengths. Again, more research will be needed to understand the basic mechanism involved.

Other nonmetallic materials (including plastic laminates, some fibrous materials, and so on) have received limited study in regard to fatigue under repeated stressing.

CONCLUDING REMARKS

A number of structural materials show a propensity to fail under repeated application of stresses lower than their static strength. Such fatigue failure is generally characterized by development of a crack at some localized weakness, subsequent propagation of the crack and ultimate weakening to a degree that renders the structural component unfit for service. The localized nature of early stages is influenced by conditions not always predictable from observed behavior of the material under steady loading.

In the case of metallic alloys, there is progress toward relating fatigue damage to the behavior of dislocations in the crystalline lattice. However, such understanding is not adequate either to understand completely fatigue damage or to deduce relations of engineering interest. In particular, cumulative damage and some interrelationships of fatigue with surface corrosion and, at elevated temperatures, with creep need more basic understanding. Meanwhile, for metals there is a rapidly growing body of data that afford some empirical rules for design and for comparative evaluation of materials. It is a challenge to the engineer to make wise use of the available information.

Less is known about the behavior of nonmetallic materials under repeated stressing. In some instances, this behavior is of direct importance in structural engineering. In general, more knowledge of the similarities and differences in fatigue behavior of metals and of nonmetals may eventually contribute toward a broader understanding of the mechanism of fatigue of materials.

It is to be hoped that the next several years will show much progress in understanding the mechanism or mechanisms of fatigue of materials. However, present knowledge indicates strongly that fatigue involves very local weaknesses and that design to prevent fatigue, even with better theoretical understanding, will require more detailed consideration than previously has been necessary for design for static strength. Such consideration of details will become more important as structures are planned to withstand higher levels of vibration and repeated stressing and, especially, as design safety factors are lowered for greater efficiency. In addition to items sketched in this brief review, the added complexities of behavior at elevated and at low temperatures, behavior under corrosive environment, effects of radiation, and effects of surface abrasion and fretting must be considered. Design to prevent fatigue of structures may thus challenge the combined efforts of civil engineers, mechanical engineers, metallurgists, physicists, and others interested in strength of materials.

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PHYSICAL METALLURGY AND MECHANICAL PROPERTIES OF MATERIALS: METALLURGICAL ADVANCES AND CIVIL ENGINEERING[a]

By Glenn Murphy,[1] F.ASCE

FOREWORD

The Engineering Mechanics Divisions Committee on Mechanical Properties of Materials conceived of the Symposium on Physical Metallurgy and Mechanical Properties of Materials as a means of summarizing, for the civil engineering profession, the current state of knowledge and some of the most recent developments in the understanding of materials behavior. The subjects of ductility, creep, brittle fracture, and fatigue have become of increasing concern in engineering applications, to the point where phenomenological descriptions of these aspects of materials behavior are becoming less than adequate for our needs.

The science of materials, stemming from physical chemistry, solid state physics, physical metallurgy, and mechanics of solids, gives us qualitative descriptions of the fundamental processes involved in the flow and fracture of solids. These fundamental concepts were presented by distinguished authorities in the 1956 Symposium on Physics of Engineering Materials organized by J. L. Waling and held at Pittsburgh, Pa. Frederick Seitz discussed imperfections in crystals, Thornton Reed, Jr., discussed dislocations, Clarence Zener discussed internal friction in metals, E. P. Blizzard and A. M. Weinberg discussed materials for radiation shielding, and G. J. Dienes discussed the effects of radiation on materials properties. The implications of these subjects in civil engineering practice were summarized by Glenn Murphy. Much of the material presented has been published in the literature of their own fields and may be found in their published works. The applications of such funda-

Note.—Discussion open until May 1, 1961. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. EM 6, December, 1960.

[a] For presentation at Symposium on Physical Metallurgy and Mechanical Properties of Materials, ASCE Annual Meeting, New York, October 17, 1958.

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mental concepts have brought about entirely new materials as well as improvement in traditional materials and indeed have created entirely new industries.

Theoretical relationships of the quantitative kind desired for engineering design purposes are, in many cases, still in the state of development. The development of such fundamental theory as exemplified by the study of brittle fracture by B. L. Averbach ("Brittle Fracture," Proceedings Paper, 2686) and the relaxation theory of creep of metals by F. H. Ree, T. Ree, and H. Eyring ("Relaxation Theory of Creep of Metals," Proceedings Paper 2333, Journal of the Engineering Mechanics Division, January, 1960, p. 41) and the importance of these concepts to engineering practice as indicated in the discussions of ductility by J. M. Frankland ("Ductility and the Strength of Metallic Structures," Proc. Paper 2687), of fatigue in structural materials by H. J. Grover ("Fatigue of Structural Materials," Proc. Paper 2688) and in the review of applications in civil engineering by Glenn Murphy ("Metallurgical Advances and Civil Engineering," Proc. Paper 2689) are considered by the committee to be invaluable to the profession of civil engineering.

In the present "Age of Materials," the problems of providing the materials suitable for the anticipated service conditions of many engineering applications require that all engineers be familiar with the latest and best of information pertaining to materials behavior. The practicing engineer must be cognizant of the rapidly growing field of materials science, which is so significant that, in some cases, it is causing large scale revisions of engineering college curricula.

The Committee on Mechanical Properties of Materials, through the 1956 Symposium on Physics of Engineering Materials, the 1958 Symposium on Physical Metallurgy and Mechanical Properties of Materials, the 1959 Symposium on the Physico-Chemical Nature of Soils, the 1960 Symposium on Nondestructive Testing of Materials, and the Symposium on Teaching of Materials in Civil Engineering Curricula being planned for 1961, have attempted to incorporate within the literature of the American Society of Civil Engineers a significant amount of the new knowledge in the field of materials behavior. Many of the participants in these symposia have been from professions other than civil engineering. A sincere expression of appreciation is hereby tendered to all who have contributed to this effort.

Joseph F. Throop, Chairman, E.M.D.
Committee on Mechanical Properties
of Materials, ASCE

SYNOPSIS

Emphasis is placed on the applicability of materials studies to all phases of civil engineering. The preceding papers of the symposium are commented on and the significance of the points raised by the authors are noted. The historical aspects of materials studies are noted as well as the need for new studies to clear up areas of some confusion.

Initially, it might be considered that a series of topics such as the ones discussed in this symposium would be of interest only to the structural engineer because of his direct concern over the relationship between mechanical properties and failure. However, further consideration makes us aware that the topics discussed are of vital significance to all civil engineers, for all are concerned with materials. Civil engineering, as the term is now commonly applied, is, in one sense, more restricted than it was at the time of the founding of ASCE over a century ago. Many fields of engineering have evolved from the civil engineering that was originally distinguished only from military engineering. On the other hand, the scope of civil engineering has expanded as new areas of learning have opened and as significant technical penetrations have been made in other areas.

The ideas discussed in the previous papers of this Symposium (Proc. Papers 2686, 2687, 2688, 2689) are not limited to one material and one use but are applicable to many of the wide variety of materials used by the civil engineer whether his particular specialization be transportation, sanitary, or structural. All must use materials, for it is through the use of materials that the ideas and the ingenuity of the engineer find tangible expression.

One of the principal occupations of the engineer is that of making predictions. His professional obligations involve making a prediction of whether a product that he is designing will be safe or whether its use will result in loss of life or property. This is his basic responsibility. If the use of the product, whether it be a bridge, an airport runway, or a water purification system, is judged to constitute a hazard, someone will have the job of redesigning it.

Down through the years diverse techniques have been used for design, for making predictions, and for insuring safety or proper functioning of a product. At one time human sacrifice was deemed essential to the success of a project. Since then we have made some progress, although we have not yet learned how to avoid sacrifice of thousands of hours of time or millions of dollars.

Our difficulties stem in part from the fact that use precedes complete understanding. We are forced to design, construct, and put into operation a wide variety of products before we understand completely how they function. In one respect this is normal because our knowledge and understanding develops from experience. A new class of phenomena, such as nuclear reactions, must usually be observed or experienced before any theory adequate for making predictions of their behavior can be formulated. In this light, if becomes evident that the degree of sophistication with which we can make predictions must be consistent with our capacity for observation and for measurement. This does not mean that we must observe a specific phenomenon before we can make a prediction regarding it, but we must know that it exists, and we must have attained, through experience stemming from observation, a respectable level of confidence regarding its general nature.

Initially, observations relating cause and effect were entirely qualitative in nature. The first builders deduced general relationships between load and failure of a beam or column. These were developed on a primitive quantitative basis as methods of measurement evolved. The rules of thumb between load and failure largely controlled structural design and the thinking of structural designers through recorded history of engineering to the time of Hooke. This approach firmly established the concept of stress as the significant criterion of impending failure since strain is less easily observed than fracture and since the accurate measurement of strain is a fairly difficult operation, it was not until about one hundred thirty years ago (about 1830) that the role of strain, even in a simple flexural member, at stress below the proportional

limit was understood. But for many years thereafter, stress was considered to be the prime criterion of impending failure.

Within recent years the importance of strain and other geometrical considerations in controlling the behavior of a material is being recognized. The recognition has come about from two sources. One is from the theoretical considerations arising through the activities of the solid state physicist, the other is through direct observation. In an attempt to explain the discrepancies between the observed strength of metals and their theoretical strength based on deduced structure, the concept of imperfections in crystal structure was introduced. Since these imperfections could be interpreted geometrically as well as in terms of potential fields, attention was turned to geometry. Improved metallographic examination techniques and the use of x-ray diffraction for probing details of atomic structure have given evidence to assist in the interpretation of the role of imperfections in structure in influencing the properties of materials.

At the same time impetus has been given to the consideration of the geometry of displacement en route to fracture (or to failure in general) through the observation of failure in situations where failure could not reasonably be explained on the basis of stress alone. Also, from a purely phenomenological viewpoint it is obvious that the criterion of failure is basically a geometrical phenomenon. We may define failure as lack of proper functioning, but this is almost invariably interpreted geometrically. The material deforms too much elastically, it behaves inelastically, it sags or droops, or comes apart. All are geometrical phenomena. The importance of geometrical considerations in making predictions of when materials will fail has been touched on several times in the preceding papers.

Ductility may be considered to be a geometrical property since it is measured in terms of a change in geometry. Frankland has brought to our attention the importance of the localized geometry upon the ductility. The capacity of a material for being distorted into a new configuration without losing continuity depends not only on its external geometry, that is its original outline and the shape into which it is to be distorted, but also on its internal geometry or crystal structure.

The influence of notches and cracks in reducing ductility, their relationship to brittle fracture, and to the reduction of the endurance limit are manifestations of the influence of external geometry. These effects are generally recognized and have been summarized for our attention today. The influence of the internal geometry is manifested at two levels at least.

One level is the microscopic, which is associated with grain size or crystal size as revealed by metallographic examination. The influence of cold working upon grain size with its attendant effect upon ductility is one example of this. The value of annealing to reduce residual stresses or to permit recrystallization has long been recognized. Once the characteristics of a given alloy have been explored by standard methods, satisfactory predictions of performance can be made.

However, in order to develop an explanation of the results attained by these empirical methods, the observational techniques must be improved. The improvement may involve consideration of smaller scale phenomena, and this leads to the study of atomic geometry or of space lattice systems and their corresponding geometrical defects. The degree of magnification required for this brings one to investigations near the forefront of research on materials. As a result of some well verified findings certain generalizations

may be made. Reference has already been made to the differences in behavior of face-centered, body-centered and hexagonal lattices.

The next step, which is not yet complete because of inherent difficulty, is that of evaluating forces and energy levels as functions of atomic structure and structural defects. Direct calculations, even for perfect crystals, are extremely tedious but machine calculations are expected to aid materially in clarifying the relationships between structure and properties.

J. M. Frankland has emphasized the fact that ductility is not an inherent property of the material, but is dependent upon the localized stress and strain. That is, the capacity of a material for distortion without failure by fracture depends on the manner in which the distortion takes place, or how the material is constrained during deformation.

Since the basic mechanism of deformation (but not the only one) is one of shear along planes of highest atomic density, it follows that the ductility of a material may be considered to denote its capacity for withstanding high shearing deformation without failure. However, the geometry of the member under consideration, the type of loading, or the orientation of the planes of probable shear may be such that no opportunity exists for the development of large scale ductile deformation. Under those circumstances the material appears to be brittle. A single crystal of metal is easily bent by hand, whereas a piece of the same size and the same material in polycrystalline form may be highly resistant to bending by hand. In the latter instance the large scale deformation is inhibited by unfavorable orientation of the many crystals, but the basic material is potentially as ductile as when it is in the form of a single crystal.

The strain concentration mentioned by Frankland is one method of measuring the boundary geometry of the member and thus serves as an index of the importance of inherent ductility in the member. As he has pointed out, a numerical association may be developed between stress raisers and the minimum safe ductility as measured by the percentage elongation in a tensile test.

The phenomenon of the deformation continuing under constant load is also closely associated with the boundary geometry of the member and the internal geometry of the material. Here it is postulated that the deformation results primarily from the movement of dislocations through the material. At temperatures approaching the melting point of some metals, the activation energy for creep has been found to be equal to that for self-diffusion. This result assists in explaining the nature of the creep process. However, at lower temperatures the two are not equal, indicating that an additional process is taking place during creep at the lower temperatures. The work of Ree and H. Eyring provides further information on the nature of the localized centers of deformation activity during creep.

B. L. Averbach has presented a comprehensive discussion of the present state of knowledge of brittle fracture. He has indicated that the failure is of a localized nature, consisting of a chain or band of localized discontinuities which propagate rapidly through the material from a point of stress concentration of strain concentration. The importance of the external geometry in providing regions of concentration has been noted. The elimination or the reduction of potency of such regions is a significant factor in design. The point has been made that the quantitative influence of surface imperfections is difficult to evaluate, being dependent upon the rate of loading, the temperature and other environmental factors and upon the material itself.

The internal geometry of the material, including the type of atomic lattice and the imperfections, ranging from grain boundaries to displaced individual atoms, is also highly significant in the detail design of an individual member. As Averbach stated, the last safety factor must be built into the metal.

It is not always possible to improve the design by adding material to reduce stress. The cost of adding material may be prohibitive in terms of weight, in reduction of flexibility, or in producing malfunction of the device. The addition of a pound of excess weight to an airplane presents certain problems, but for a satellite or missile the addition of weight that is not absolutely necessary results in design and operational problems that are far more serious.

Some of the problems of developing satisfactory and reliable tests of materials to serve as a starting point for design have been discussed, and their influence on specifications noted. In such tests of materials for specifications at least three dangers are always present. One is whether or not the material in the specimen tested is truly representative of the material to be used or whether the variations between lots or melts of material bias the results. There are many examples of difficulties arising from differences among samples.

Another danger comes from the implicit assumption that the significant property in the material for the given application correlates with the property evaluated by the test. The resistance of a material to load is the resultant of sevdral component resistances, and the balance among them is often delicate. The component resistances are structure sensitive and sensitive to environment. Hence, a difference of a few degrees in temperature may shift the balance among the component resistances and lead to false indications from the test specimen.

The third factor is the effect of surface treatment or condition. The authors have stressed the effects of cracks, notches, and other surface irregularities. A difference between the surface preparation of the part to be used and the test specimen may invalidate the results from the test.

Thus, we are led to the conclusion that tests to determine whether or not a given material meets a specification must be interpreted in the light that meeting the specification does not insure satisfactory performance. It only means that the sample tested meets a test requirement that experience has shown to be a satisfactory index of performance in the product. This point of view is satisfactory for an application where ample experience has been gained, as in ordinary mild steel reinforcing bars for concrete. However, it is not satisfactory for new applications as has been shown many times. In a new application the required balance between contributing resistances may be different than in any previous application, and the cost of making many trials to gain the necessary experience may be prohibitive. In determining the worth of a new material or the significant requirements of the material in a new application, a fundamental approach must be used, and this is gained through analysis and study of the problem. This point of view has been implicit in the remarks of all four of the authors of the papers of this Symposium.

H. J. Grover, in particular, has pointed out that in the consideration of fatigue, attention to small detail is necessary to a greater degree than in the consideration of steady stress. He has noted that the development of a theory of fatigue is normally based on the assumption that failure starts at a small discontinuity in the material as a result of stress of strain concentration and that the resulting crack works across the material as the stress is removed and reapplied. The fact that unloading and reapplication of load is necessary

for the propagation of the crack is evidence of a change in the internal geometry of the material as a result of unloading. This has led to a consideration of the possible mechanism of movement of dislocations or other imperfections during unloading. One tentative conclusion is that the mechanism is somewhat like that of a rachet in permitting motion of the dislocations in only one direction. This is closely associated with the problem of cumulative damage under repeated loading.

Experiments on the behavior of uranium under repeated loading at certain stress levels have revealed the formation of an extensive network of surface cracks after a relatively few cycles of loading and this may be followed by millions of cycles of loading before any crack or series of cracks progresses through the specimen. Thus, the formation of a crack does not necessarily indicate impending failure in fatigue. Here again, the seriousness of a crack, or the influence of the external geometry depends upon the nature of the material.

For some materials, fatigue cracks developed in one stress and temperature range appear within the individual grains, while in other ranges of stress and temperature the cracks are intergranular. From this it may be concluded that fatigue failure may involve more than one mechanism, and that more than one strength criterion may be necessary to make possible the prediction of the endurance limit of a new alloy or the influence of a change in environment on resistance to repeated loading.

Long ago we learned that the ultimate tensile strength was not enough to describe completely the characteristics of a material for load-carrying purposes. This fact is sometimes recognized in specifications. We learned that the deformation characteristics are also highly significant in many applications, and the authors have shown the effect of certain environmental factors upon those characteristics. In so doing, they have emphasized the fact that the properties of a material fall into two classes—those which are essentially unchanged by changes in environment and those which are markedly affected by environment and by internal geometry or structure. Those properties associated with resistance to fracture or to inelastic action under load are structure sensitive, in general.

Thus, so far as strength is concerned, it is dangerous to think in terms of an inherent resistance of a given material, for the resistance or the combination of resistances varies with conditions. We may think in terms of a potential strength of the material in a given environment and under a given load or change in configuration, but the available strength in materials as normally fabricated is an entirely different quantity.

We must not lose sight of the question of what properties we really want in materials. We obtain talk of strength-weight ratios and are inclined to use them as figures of merit, but such figures, usually based on axial loading may be misleading in an actual application. The tensile strength of water is practically zero as normally interpreted, but under equal triaxial tensile stress it is high.

The various arts of increasing the axial tensile strength of a metal normally result in decreased ductility. It is clearly apparent that a gain in strength at the expense of ductility, or capacity for geometrical adjustment under load, is not always desirable.

Many years ago, this need for "ductile" adjustment formed the basis of Kipling's, "Ship that Found Herself." In the famous "One Hoss Shay" we have the picture of the ultimate fate of a product designed on the basis of strength.

The result came with the suddenness of a fatigue failure and the finality of brittle fracture.

Another item that has not been particularly emphasized in the discussions but which has a bearing on the entire problem is that of the statistical nature of materials. The mechanics that we, as civil engineers, are accustomed to use in calculating stresses and strains is Newtonian mechanics, based on Newton's laws of motion for a single particle and extended to rigid bodies. However, at the level of magnification that the strength properties of a material are actually established, the material cannot be considered as a rigid continuous medium. It must be regarded as a seething mass of electrons and atomic nuclei in constant motion and with continually varying interatomic forces that will control the properties. Thus, the properties as measured in a test must be regarded as statistical, and the qualities required in the piece as used are also statistical because of uncertainties in loading.

Hence, we find that the problem of predicting performance must, in the last analysis, also contain an indication of the desired or required probability of the prediction. The adoption of a statistical point of view is inevitable in our exploration for more reliable methods of reducing the failure of materials in service. On a phenomenological scale it will be needed by the designer in establishing limits of confidence in predictions, on a microscopic scale it will be needed by the metallurgist and the physical tester in controlling and recording the properties of materials, and on the atomic scale it will be needed by physicist in bbringing us nearer to the day when materials, like structures, will be designed to perform as desired.

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ELASTO-PLASTIC ANALYSIS BY NUMERICAL PROCEDURES

By Annabel L. Tong,[1] M. ASCE

SYNOPSIS

An elasto-plastic analysis for the bending and buckling of prismatic structural members by numerical procedures is presented in this paper. A stress-strain diagram composed of two straight-line segments is used to approximate the actual stress-strain diagram of the material. This simplifies the computations and gives reasonably good accuracy for materials having defined stress-strain diagrams.

INTRODUCTION

A numerical analysis for structural members subjected to bending and buckling in the elasto-plastic range is presented. Two general assumptions are made in this analysis: first, that the material of the member is homogeneous and plane sections remain plane; second, that the history of loading is assumed to have no stress reversal and that there is no appreciable initial residual stress in the members under investigation.

The method described was developed several years ago in a thesis by the author.[2] It is summarized here to illustrate its application to pure bending and buckling problems with a hope that this method can be generalized and possibly

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[1] Struct. and Hydr. Engr., Stearns and Wheler, Cons. Engrs., Cazenovia, N. Y.

[2] "A Study on Column Analysis," by Annabel Y. W. Lee, Thesis, Presented to Cornell, Univ., at Ithaca, N. Y., in 1949, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

refined to solve other more complicated design problems beyond the elastic limit. A simple stress-strain diagram composed of two straight lines with different slopes is recommended to replace the actual stress-strain diagram of a material. For easy illustrations, members of rectangular sections are used in the examples and the material is assumed to have identical stress-strain relationship in tension and in compression. This simplified diagram is sufficiently accurate for metals such as some aluminum alloys, cold-rolled steel and high-strength steel. In Fig. 5, a typical stress-strain diagram for an aluminum alloy is shown by two straight-line segments of slopes E_1 and E_2 that approximate its actual stress-strain relationship. A curve showing the change of tangent-modulus with respect to the stresses is also plotted on the same graph for comparison.

Notation.—The letter symbols adopted for use in this paper are defined and arranged alphabetically, for convenience of reference, in the Appendix.

PURE BENDING OF BEAMS

The bending theory adopted in this analysis is based on the principle of flexure used in the classical elastic theory for pure bending of beams. The methods of moment-area and conjugate-beam are applicable here, but the method of superposition to determine stress or strain in beams under several loads will not be true in the elasto-plastic range.

The simplified stress-strain diagram shown in Fig. 1 represents the general relationship between stresses and strains in tension and in compression for materials having a sharply defined yield point. For practical purposes, it is reasonably accurate to assume that such a material would have a constant modulus of elasticity E_1 below the yield point and a different modulus E_2 above the yield point. By the flexural principle, the unit fiber strain is directly proportional to the distance of the fiber from the neutral axis. A beam of symmetrical cross section subjected to bending will have its stress distribution and corresponding strain diagram at its critical section as shown in Fig. 2. When there is no axial load at a section, the fiber stress distribution must satisfy

$$\int_{o}^{c} f\ w\ dy + \int_{-c'}^{o} f'\ w\ dy' = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

and

$$\int_{o}^{c} f\ w\ y\ dy + \int_{-c'}^{o} f'\ w\ y'\ dy' = M \quad \ldots\ldots\ldots\ldots (2)$$

Assume that the tensile side of the f-e diagram is identical to the compression side. Then $c' = c$, $y = e\ r$, and $c = e_m r$. For rectangular sections w is a constant and Eq. 2 becomes:

$$M = \int_{-c}^{c} f\ w\ y\ dy = 2\ w\ r^2 \int_{o}^{e_m} f\ e\ de = \frac{I}{r}\ \frac{3}{e_m^3} \int_{o}^{e_m} f\ e\ de = \frac{\bar{E}\ I}{r} \quad \ldots . (3)$$

in which

$$\bar{E} = \frac{3}{e_m^3} \int_{o}^{e_m} f\ e\ de \quad \ldots\ldots\ldots\ldots\ldots (4)$$

which will be called the effective modulus. Substituting μ for E_2/E_1 and α for e_m/e_y, transforms Eq. 3a into Eq. 5.

$$\bar{E} = \frac{E_1 e_y}{e_m}\left[(1-\mu)\left(\frac{3}{2} - \frac{1}{2\alpha^2}\right) + \alpha\mu\right] = \frac{M r}{I} \quad \ldots (5)$$

Eq. 5 can be written as

$$\alpha^3 + \alpha^2 (3B - A) - B = 0 \quad \ldots\ldots\ldots\ldots (6)$$

in which

$$A = \frac{Mc}{I E_1 e_y \mu} = \frac{M}{M_y \mu} \quad \ldots\ldots\ldots\ldots (7)$$

and

$$B = \frac{1-\mu}{2\mu} \quad \ldots\ldots\ldots\ldots (8)$$

Fig. 3 shows a set of "α" versus "A" curves for rectangular beams of materials with B values ranging from 100 to 10 as computed according to Eq. 6. The term B is a constant for a given material when E_1 and E_2 are determined. The value A at any section can be computed when the moment at that section is known. With the value of α from Fig. 3, the effective modulus $\bar{E}$ and the maximum fiber stress f_m are given by

$$\bar{E} = \frac{Mc}{I e_y \alpha} \quad \ldots\ldots\ldots\ldots (9a)$$

$$\frac{M}{\bar{E} I} = \frac{\alpha M_y}{E_1 I} \quad \ldots\ldots\ldots\ldots (9b)$$

and

$$f_m = f_y + E_2 e_y (\alpha - 1) \quad \ldots\ldots\ldots\ldots (10)$$

Knowing $\bar{E}$, the $M/\bar{E}I$ diagram could be drawn along the entire length of the beam, from which the slope changes and deflection curve can be computed by any of the methods used in the conventional elastic analysis such as conjugate-beam, moment-area, or graphical method. For most engineering materials, E_2 is small compared to E_1; hence when a beam is loaded not too far beyond the elastic limit, one may assume $E_2 = 0$, or, $\mu = 0$. Eq. 6 then becomes

$$\alpha = \frac{1}{\sqrt{3 - \frac{2M}{M_y}}} \quad \ldots\ldots\ldots\ldots (11)$$

It is seen that as M reaches 1.5 M_y, α and e_m both approach infinity. This moment M_p is usually called the ultimate moment in the plastic design. Under

this moment a plastic hinge is produced at the critical section of the beam and the parts on either side of the section rotate relatively to one another while the section transmits a constant moment equal to the ultimate moment.

Example 1.—A beam of rectangular section is simply supported at both ends with a load P acting at the center of the span (Fig. 4). The maximum stress and deflection of the beam at its midspan are determined as follows:

Using the stress-strain diagram shown in Fig. 5 for illustration, and assuming the dimensions of the beam as to be L = 48 in., w = 1 in., and h = 2 in., one obtains that M_y = 32,700 in. - lb and B = 20 for this beam. Then the $M/\overline{E}I$

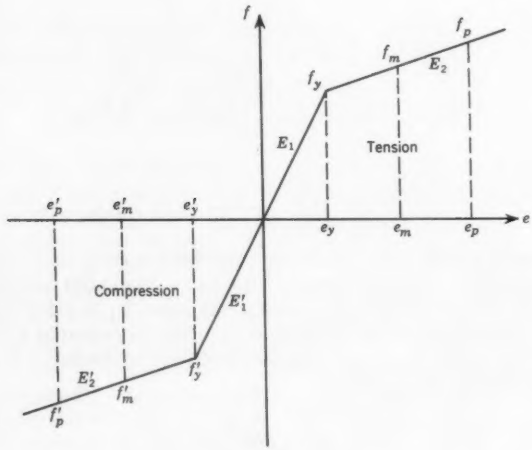


FIG. 1

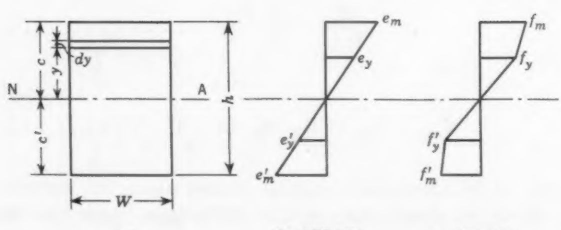


FIG. 2

diagram can be plotted by dividing the lengths of the beam into intervals of λ. To find θ_A and θ_B at the supports and the deflection of the beam, Newmark's[3] numerical procedure can be used as illustrated in Fig. 4. From Fig. 4

$$\theta_A = \theta_B = \frac{100}{E_1 I}\left(\frac{\lambda}{12}\right)(36{,}240 + 288) = 0.266 \text{ radians}$$

[3] "Numerical Procedure for Computing Deflections, Moment and Buckling Loads," by N. M. Newmark, Transactions, ASCE, Vol. 108, 1934.

$$\text{Center deflection } y = \frac{100}{E_1 I}\left(\frac{\lambda^2}{12}\right)(122{,}840) = 5.23 \text{ in.}$$

and

$$\text{Maximum Stress } f_m \text{ at mid-span} = 62{,}700 \text{ psi.}$$

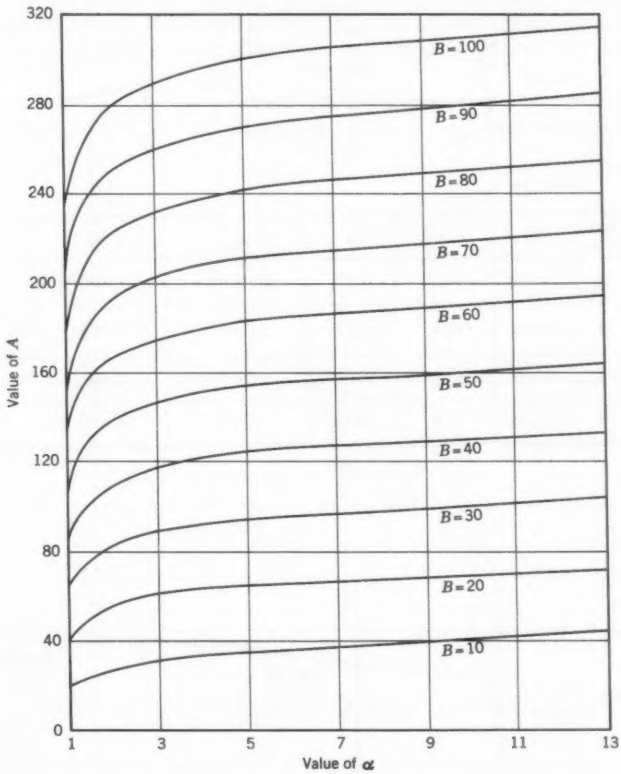
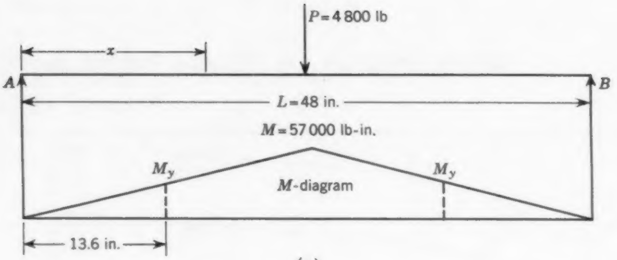


FIG. 3

Assuming the ultimate stress f_p to be 1.33 f_y, the ultimate load is determined by solving for α in Eq. 10, or

$$1.33\, f_y = f_y + E_2\, e_y\, (\alpha - 1) \quad \ldots\ldots\ldots\ldots\ldots (12)$$

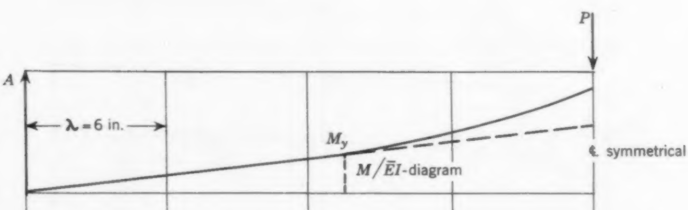
In this example, α is found to be 14.6. Fig. 3 gives the corresponding A value to be 74.6. Therefore, the ultimate moment M_p is μ A M_y or 59,300 in. - lb and the ultimate load is 4950 lb. From the elastic analysis the yielding load allowed to produce maximum stress in the beam equal to the yielding stress, is 2720 lb. A beam as shown in Fig. 4 designed for a working load based on



(a)

x, in inches	M	A	α	$\bar{E}=\frac{Mc}{Ie_y\alpha}$	$M/\bar{E}I$
15	36 000	45.3	1.22	9.3×10^6	$40000/E_1I$
18	43 100	54.4	1.66	8.2×10^6	$54000/E_1I$
20	48 000	60.4	2.85	5.3×10^6	$93500/E_1I$
24	57 600	72.5	12.60	1.5×10^6	$412000/E_1I$

(b)



										Common factors
$M/\bar{E}I$	0		144		288		540		4 120	$100/E_1I$
Equiv. conc. ld.	−288		−1 728		−3 564		−9 808		−42 280	$(100/E_1I)\lambda/12$
Slope		36 240		34 512		30 948		21 140		do.
Deflection	0		36 240		70 752		101 700		122 840	$(100/E_1I)\lambda^2/12$

(c)

FIG. 4

its ultimate load divided by a safety factor less than 1.82 (the ratio of the ultimate load to the yielding load) will be subjected to stresses beyond the elastic limit at the center portion of the beam. To determine the stress or deformation of the beam so loaded, the effective modulus $\bar{E}$ should be used to construct the $M/\bar{E}I$ diagram instead of the elastic modulus E_1.

Eq. 6 and Fig. 3 are derived for rectangular beams only. For cross sectional shapes other than rectangles, the same derivation procedure would produce similar results. For framed structures where the carry-over factor and the relative stiffness of a member vary with the amount of moment distributed to the member, as well as with other factors such as E, I, and L of the member,

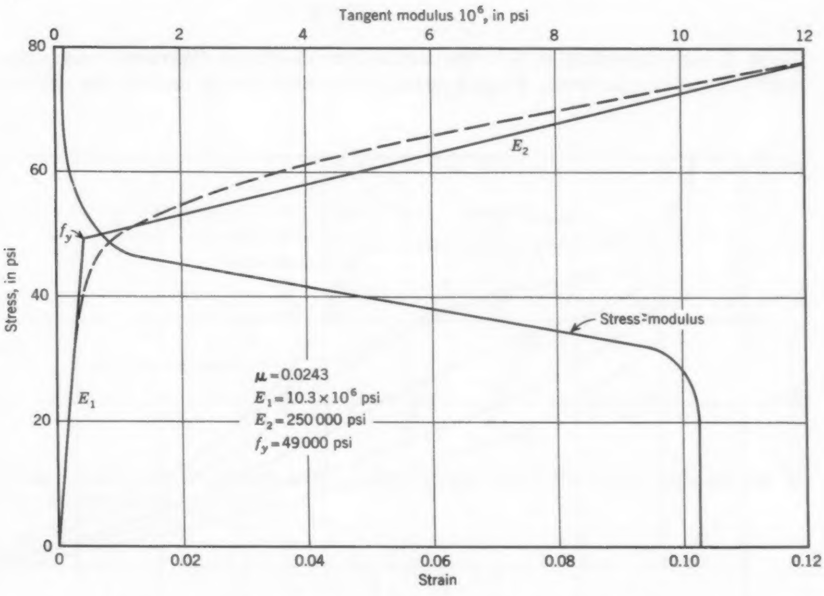


FIG. 5.—STRESS—STRAIN DIAGRAM OF ALUMINUM ALLOY

methods of successive approximation[4] must be employed to calculate the final moments under elasto-plastic loadings.

BUCKLING OF COLUMNS

The buckling equation for a perfect column derived from the simplified stress-strain diagram (Fig. 5) is:

$$f_{cr} = \frac{\pi^2 E}{(L/R)^2} \qquad (13)$$

in which $E = E_1$ for $f_{cr} < f_y$ and $E = E_2$ for $f_{cr} > f_y$.

4 "Elastic-Plastic Analysis of Continuous Frames and Beams," by Lawrence P. Johnson, Jr. and Herbert A. Sawyer, Jr., Proceedings, ASCE, Vol. 84, No. ST 8, December, 1958.

The buckling curves for axially loaded columns plotted in Fig. 6 for $\epsilon c/R^2 = 0$ are derived from Eq. 13. It is seen that the portion of this column curve below the yield stress coincides with the Euler's curve. Because of the discrepancy between the actual and the simplified diagrams, this curve gives higher critical stresses near the yield point than the values obtained by the tangent modulus formula[5] which is derived from the actual stress-strain diagram of the material. The Tangent-Modulus curve for the perfect column derived from the formula expressed by:

$$f_t = \pi^2 \quad \frac{E_t}{(L/R)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(14)}$$

(E_t is E corresponding to f_t in the actual stress-strain diagram), was plotted in Fig. 6 for comparison. When a column is eccentrically loaded, the effect of

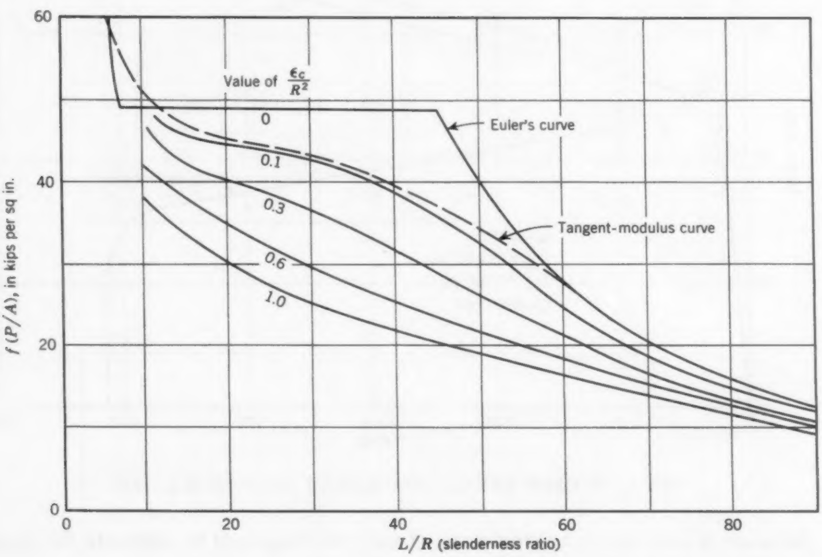


FIG. 6.—COLUMN CURVES FOR ECCENTRICALLY LOADED COLUMNS

this discrepancy seems to be minimized and good approximation can be obtained with the simple computations required by this method as compared to other more exact methods.

The method of investigating the ultimate strength of eccentrically loaded columns is outlined by the procedure illustrated in Example 2.

Example 2.—A rectangular pin-ended column AB (Fig. 7) with uniform depth h and width w is assumed to be made of the material having the stress-strain relationship shown in Fig. 5. When this column is loaded beyond the elastic

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[5] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Company, Inc., New York, N. Y., 1936.

limit by a compressive force P with eccentricity ϵ, the stress distribution over the cross section at mid-span will be of one of the cases shown in Figs. 8, 9, and 10 with the assumption that the column remains nearly straight until buckling occurs. Let $\bar{f} = \frac{P}{w\,h}$ = Average compressive stress; e_0 = Strain of the center fiber; e_1 = Maximum tensile strain; e_2 = Maximum compression strain; and $\Delta = e_1 + e_2 = h/r$. Hence

$$y = r(e - e_o) \qquad (15a)$$

$$e_o = (b - a)\, e_y/2 \qquad (15b)$$

$$a = (e_1 - e_y)/e_y \qquad (15c)$$

and

$$b = (e_2 - e_y)/e_y \qquad (15d)$$

In case 1, $e_1 > e_y$ and $e_2 > e_y$, Eqs. 1 and 2 become:

$$\bar{f} = \frac{1}{w\,h} \int_A f\, dA = \frac{1}{\Delta} \int_{e_1}^{e_2} f\, de \qquad (16)$$

and

$$M = \int_A f\, y\, dA = \frac{12\, I}{\Delta^2\, r} \int_{e_1}^{e_2} f\, (e - e_o)\, de \qquad (17)$$

Using + f for tensile stress and - f for compressive stress in Eqs. 16 and 17, Eqs. 18 and 19 are obtained.

$$\frac{\bar{f}}{f_y} = \frac{b - a}{a + b + 2} \left[1 + \frac{\mu}{2} (a + b) \right] \qquad (18)$$

and

$$\frac{M}{M_y} = \frac{6}{(a + b + 2)^2} \left\{ \frac{2}{3} + (a + a\,b + b) + \frac{\mu}{2} \left[\frac{a^3 + b^3}{6} + a^2 + b^2 + \frac{a\,b}{2} (a + b) \right] \right\} \qquad (19)$$

In case 2, $e_1 < e_y$ and $e_2 > e_y$, then

$$\frac{\bar{f}}{f_y} = \frac{1}{a + b + 2} \left[b \left(1 + \frac{\mu\, b}{2} \right) - \frac{a}{2} (a + 2) \right] \qquad (20)$$

and

$$\frac{M}{M_y} = \frac{6}{(a + b + 2)^2}\left[\frac{1}{3} + \frac{(a + 1)^3}{3} + \frac{a\,(b-a)\,(a+2)}{4}\right.$$

$$\left. + \; b\left(\frac{a}{2} + 1\right) + \frac{\mu\, b^2}{2}\,(b + 3\,a + 6)\right] \ldots\ldots\ldots \quad (21)$$

Eqs. 20 and 21 will also cover case 3 as shown in Fig. 10 where the compressive stress acts over the entire section, that is, $-2 < a \leq -1$.

FIG. 7

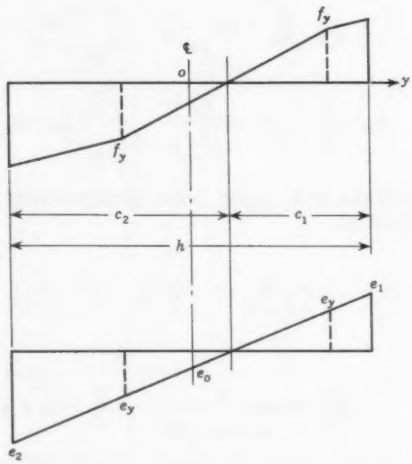

FIG. 8

From Eqs. 18 and 20, values of $\bar{f}/f_y$ can be plotted against a and b in a family of curves as shown in Fig. 11. From Eqs. 19 and 21 and with the curves in Fig. 11, values of $\bar{f}/f_y$ can be plotted for various M and Δ values in another group of curves as shown in Fig. 12. As an example, Fig. 12 is prepared non-dimensionally for rectangular columns of material with $\mu = 0.0243$.

Once the M-Δ diagram for columns of certain material and known cross section is established, the deflections at all sections can be determined for a column under compression P by assuming Δ for each section. The deflection curve then

could be plotted by a semi-graphical procedure as follows:

1. Assume Δ_o at the midspan of a column under compression P.

2. Determine M_o from the M-Δ diagram (Fig. 12) corresponding to the given P and assumed Δ_o.

3. Compute $\delta_o = M_o/P$ as the deflection at midspan of the column measured from the line of action of P, then compute $r_o = h/\Delta_o$ as the radius of curvature at midspan.

4. Compute the off-set values, d_1, d_2, d_3 ----, of a deflection curve with a maximum value δ_o at the center of the span. The computing procedure is illustrated in Table 1 where "s" is a small segment of the column. The complete

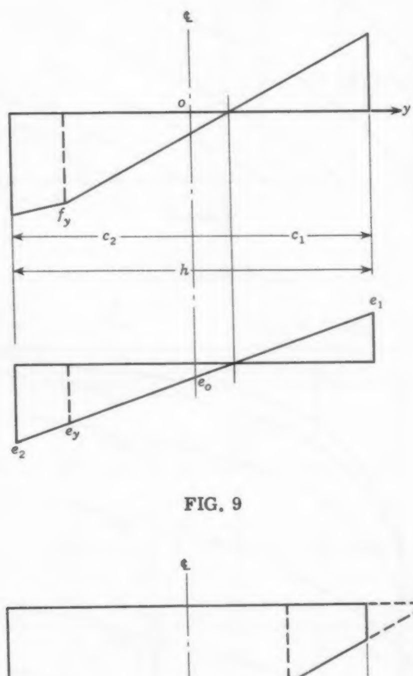

FIG. 9

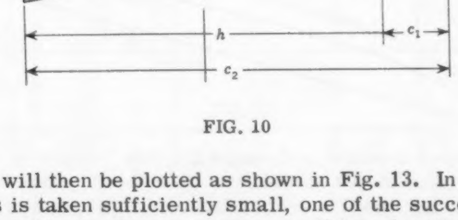

FIG. 10

deflection curve will then be plotted as shown in Fig. 13. In the course of this computation, if s is taken sufficiently small, one of the successively computed d_n will become equal to δ_o. In most cases, n will be a fractional number rather than an integer in order to satisfy $d_n = \delta_o$. It is seen from Fig. 13 that a column

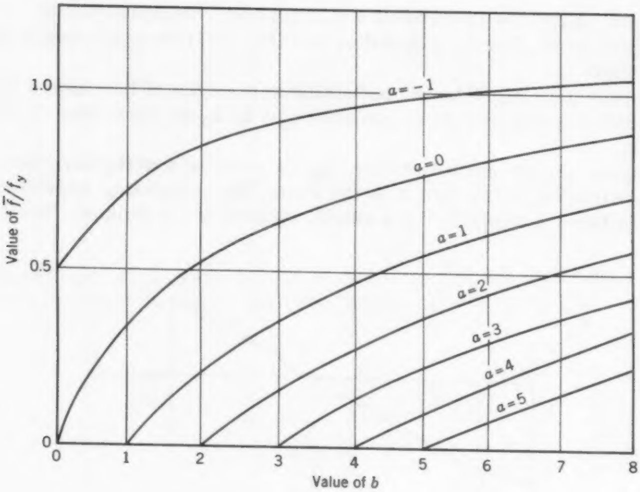

FIG. 11

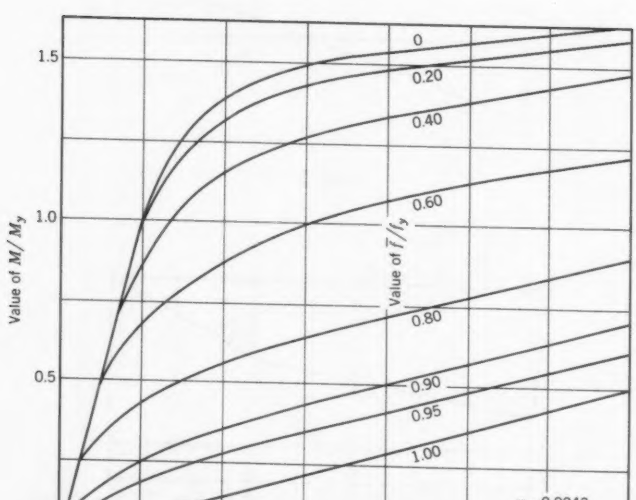

FIG. 12.—NON-DIMENSIONAL M-Δ DIAGRAM

of half-length equal to n s is in equilibrium under a concentrically applied load P, and a pre-selected midspan deflection δ_o.

5. From a curve thus obtained (Fig. 14 (a)) it is observed that the central portion of the column of length L represents an eccentrically loaded column with eccentricity ϵ_1 which has the same configuration in equilibrium as the

TABLE 1

d	M	Δ	r
$d_1 = \frac{s^2}{2\,r_o}$	$M_1 = P\,(\delta_o - d_1)$	Δ_1	$r_1 = \frac{h}{\Delta_1}$
$d_2 = d_1 + \frac{s^2}{r_o} + \frac{s^2}{2\,r_1}$	$M_2 = P\,(\delta_o - d_2)$	Δ_2	$r_2 = \frac{h}{\Delta_2}$
$d_3 = d_2 + \frac{s^2}{r_o} + \frac{s^2}{r_1} + \frac{s^2}{2\,r_2}$	$M_3 = P\,(\delta_o - d_3)$	Δ_3	$r_3 = \frac{h}{\Delta_3}$
....	....	...	....
$d_n = d_{n-1} - - - \frac{s^2}{r_{n-2}} + \frac{s^2}{2\,r_{n-1}}$	$M_n = P\,(\delta_o - d_n)$	...	....

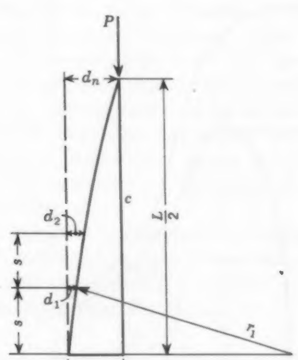

FIG. 13

original column of length L and axial load P_1. Hence a single deflection curve computed as previously described, represents the equilibrium configurations of an entire set of columns under axial force P_1 with different eccentricities ranges from zero to δ_o.

6. Using a given value of $\overline{f}_1$ (P_1/A) and various mid-span deflections δ_o, a set of curves for L versus δ_o can be obtained for various ϵ values as shown in Fig. 15. After determining the L - δ_o curve for an eccentricity ϵ_1, the maximum point Q_1 of the curve gives the maximum length of an eccentrically loaded

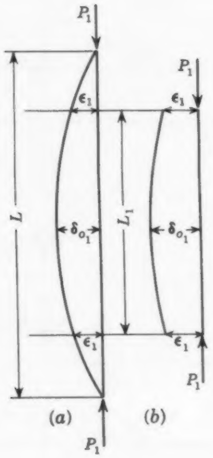

FIG. 14

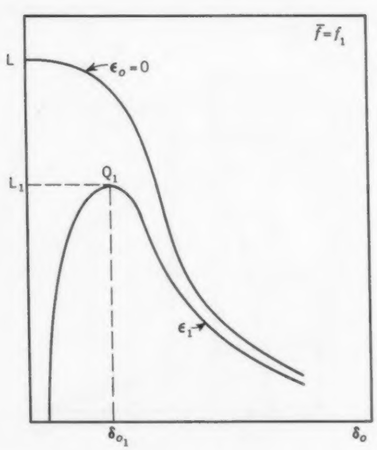

FIG. 15

column of ϵ_1 and $\overline{f}_1$, which is also the critical length of this column in stable condition. The column curve of $\overline{f}$ versus L/R for various values of equivalent eccentricity $\epsilon c/R^2$ can be drawn after the critical lengths of columns are determined by the maximum points from the graphs plotted for various $\overline{f}$ values such as the one shown in Fig. 15.

Fig. 6 is an example of the column curves drawn by the preceding procedure for materials having a simplified stress-strain diagram given in Fig. 5. This method gives an exact solution of the buckling strength for columns of materials having f-e diagram composed of two straight lines and is a good approximation for materials whose f-e diagram has a sharply defined yield point. This procedure is in the reverse of, but achieves the same result as, the customary methods used in elastic analysis. In the method presented, the length of a column is calculated to satisfy the equilibrium condition for the given load and a pre-selected deflection. This approach lends itself very well to the analysis of eccentrically loaded inelastic columns, and it is this feature that is of value.

The buckling loads thus obtained are for the hinged columns. For columns with end restraints, an effective length would be used in place of the actual length of the column as is customary in elastic analyses. For columns of cross sections other than rectangle, a shape factor "k" could be applied to $\epsilon c/R^2$ so that the equivalent eccentricities expressed by $k(\epsilon c/R^2)$ would replace the original $\epsilon c/R^2$ values. With the rectangular shape as the reference cross section, that is, $k = 1$, a set of generalized column curves can be prepared for practical designs of various columns with or without restraints.

CONCLUSIONS

The representation of the stress-strain diagram for structural materials by two straight lines of different slopes results in considerable simplification to structural problems beyond the elastic limit. In the determination of the ultimate strength of columns, the method presented utilizing this simplified stress-strain diagram, is still time consuming but not impractical as demonstrated by the given examples. This simplification applied to the determination of critical loads of perfect columns shows large discrepancy in the range near the yield point stress (Fig. 6) but it is sufficiently accurate for determining critical loads of eccentrically loaded columns. The column curves shown in Fig. 6 are derived by a semi-numerical and semi-graphical method in which the critical length of a column is determined with a given load and a pre-selected deflection. The advantage of this method lies in the fact that families of graphs for various loadings and eccentricities are customarily required in practical column design so that the time and effort spent in their preparation would be justified by their usefulness for designing columns of the same material under various typical loading conditions.

The examples given are limited to members of rectangular sections for easy illustration. Members of other cross sectional shapes will have different A versus α (Fig. 3) curves and M versus Δ (Fig. 12) curves, but the same procedure would be used with little modifications.

APPENDIX. NOTATIONS

---

c, c' = distances of extreme fibers from neutral axis;

$\bar{E}$ = effective modulus;

E_1 = average modulus of elasticity below yield point stress;

E_2 = average modulus of elasticity beyond yield point stress;

E_t = tangent modulus of elasticity;

e = unit fiber strain;

e_m = maximum fiber strain;

e_p = ultimate strain;

e_y = yield point strain;

f = unit stress;

f_m = maximum fiber stress;

f_p = ultimate strain;

f_y = yielding stress;

h = depth of beam;

I = moment of inertia;

L/R = slenderness ratio;

M = bending moment;

M_p = ultimate moment;

M_y = yielding moment;

R = radius of gyration of the cross section;

r = radius of curvature at neutral axis;

w = width of beam;

α = e_m/e_y

Δ = total strain;

ϵ = eccentricity; and

μ = E_2/E_1.

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ENGINEERING MECHANICS DIVISION

Proceedings of the American Society of Civil Engineers

DISCUSSION

Note.—This is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. EM 6, December, 1960.

ELECTRICAL ANALOG FOR EARTHQUAKE YIELD SPECTRA[a]

Closure by G. N. Bycroft, M. J. Murphy and K. J. Brown

G. N. BYCROFT,[1] M. J. MURPHY,[2] AND K. J. BROWN.[3]—The authors appreciate the kind remarks made by the discussers to this paper. The extremely painstaking work by Messrs. Berg and Thomaides (which we have had the opportunity of studying in more detail than appeared in the discussion) has been most valuable, serving both as a confirmation and correction of our own work: confirmation, in that the measure of earthquake energy dissipation in a structure has been obtained by two sets of investigators and found to be within limits that the practising engineer can accept - thereby offering hope of a consistent design procedure for earthquake structures; correction, in that differences as large as $2\frac{1}{2}$ to 1 between his figures and ours have been obtained. It is certain that the process of digital computation is more accurate than an analog procedure and Mr. Berg's comprehensive check procedure would seem to eliminate possibility of error. At only one point can doubt possibly exist and that is the alteration of the accelerogram from its original form. We, like Mr. Berg, altered our accelerogram. In our case, the base line was bent to give zero terminal velocity of the ground. On reflection we now think that any alteration of the accelerogram is invalid. It seems now that such an alteration is unnecessary if it does not materially affect the results and philosophically unacceptable if it does. Having said this, we must admit that we do not think that this explains very much of the disparity between our results.

Attention then is focussed on our analog techniques. Mr. Housner has suggested a frequency effect in our yield circuit. Pictures of the yield characteristic up to 10 kc (the highest frequency involved) are shown in Fig. 1. I think the yield characteristic is sufficiently straightsided to dispose of this as a source of serious error. It seems that the major disparity between the digital and analog methods occurs where the yield deflections are small compared to the elastic deflections. This occurs at long periods and at high levels of yield acceleration. It is also in this condition that the analog technique we used is most suspect. It is some saving grace that this condition is the least significant to the engineer. Nonetheless, it is apparent that our techniques have not been sufficiently accurate and they are being overhauled.

At this stage of enquiry into a new field we think it is best to summarize the position by saying that Mr. Berg's data is probably more correct than ours and wait for further work to define the position more clearly.

---

[a] October, 1959, by G. N. Bycroft, M. J. Murphy, and K. J. Brown.

[1] Research Officer, Bldg. Physics Section, Div. of Bldg. Research, National Research Council, Ottawa, Canada.

[2] Chief, Mech. Engr., Dominion Physical Lab., Lower Hutt, New Zealand.

[3] Tech. Officer, Engr. Seismology Section, Dominion Physical Lab., Lower Hutt, New Zealand.

Mr. Glogau has asked that we correct his statement that the "Polytechnical School in Mexico City appears to have been a dynamic design." A later contribution from Mr. Binder, Chief Engineer, Bethlehem Steel Corp., advises that this building was of short period, did not exhibit any foundation failure, and did not reflect a proper test of a seismically-designed structure. Mr. Glogau's other points may be summarized by the statement that rigid buildings may well be able to use inherent damping and plastic yield qualities of the foundation better than a flexible structure. This is true and there is some slight evidence to support the belief that this actually occurs. Unfortunately no one has yet measured the dynamic and plastic properties of soils. It would seem to be a worthwhile project for someone to undertake.

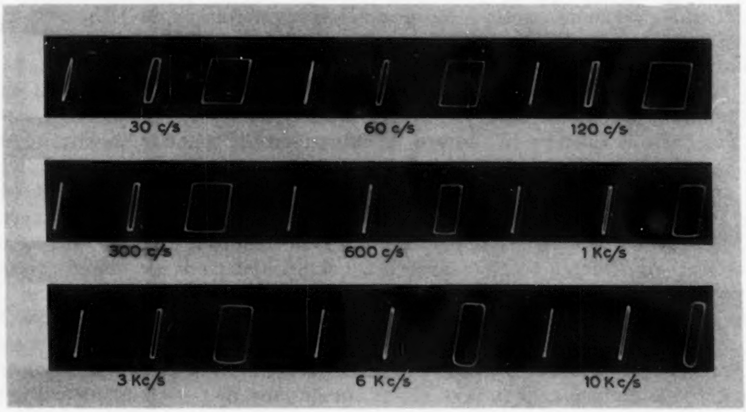

FIG. 1

That there will be a general relationship between the energy built up in a fully elastic structure and the energy dissipated in an elasto-plastic structure (as has been put forward by Mr. Blume) is to be expected. A design procedure which relates the energy of the elastic spectra with the area under the static load deflection curve to failure seems to err overmuch on the side of safety. It would be reasonable to suppose that excursions to and fro at amplitudes less than this would enclose some form of hysteresis loop. None the less, at the present level of knowledge in this subject, Mr. Blume's procedure is a considerable improvement on current practice. The elastic curve (Fig. 16) commented on by Mr. Blume contains a calibration error and is too low by some 25%.

LATERALLY LOADED THIN FLAT PLATES[a]

Closure by William A. Bradley

WILLIAM A. BRADLEY,[24] M. ASCE.—As stated in the original paper, the writer agrees with Messrs. McInnis, Tsai, and Sims regarding certain advantages of using metal models in the Moiré method, because errors may be introduced by the use of the calibration plate for determination of physical constants. In cases where clamped edges are desired, however, it is felt that the plastic may still be preferable, because its low modulus of elasticity - less than one-tenth that of magnesium - will result in less elastic deformation of the support itself and thus a closer approximation to true fixity.

TABLE 1

	$\dfrac{W_{center}}{q\,a^4/D}$	Max. edge $\dfrac{M}{q\,a^2}$
Grid spacings:		
a/4	.0017995	-.038624
a/6	.0015344	-.044728
a/8	.0014244	-.047368
a/10	.0013696	-.048715
Extrapolations:		
8,10	.0012722	-.051110
4,6,8	.0012700	-.051148
6,8,10	.0012661	-.051305
4,6,8,10	.0012653	-.051335

It is suggested that the Moiré method may give better results, in some cases, than the finite difference method; this is possible, although with the experimental method, it is impossible to know the percentage of error or on which side of the true value it lies. With finite differences, by using successively finer grids, convergence toward the true value will usually result, indicating the direction of the error. Also, by the use of extrapolations, greatly improved results can be obtained. These, in certain cases, will be "exact."

For the square clamped plate with uniform load, results from different grids were shown in Fig. 14. Similar results given to higher place accuracy and with extrapolations are shown in Table 1.

---

[a] October, 1959, by William A. Bradley.

[24] Assoc. Prof., Applied Mechs., Michigan State Univ., East Lansing, Mich.

The center deflections of .0012653 qa^4/D compares with the value of .0012763 obtained by Mr. McInnis, et al., using the Rayleigh-Ritz method and that of .0012653 obtained by Wojtaszak. In plates of unusual shape, loading or boundary condition, the experimental method might lead to closer approximations than would be obtained by a difference solution which was practical from a standpoint of human or machine computer time.

The method of loading mentioned by Mr. Terzian is of interest and might be particularly useful in the application of loads of varying intensity. Mr. Terzian also referred to the alinement procedure as followed; the model, the grid and the negative planes were all adjusted first in a vertical plane by the use of a spirit level. After the vertical adjustment, the planes were then made parallel horizontally using direct measurements to reference points in the three planes. No computations have been made as to the possible error caused by failure of these three elements to be parallel, since it was felt that they were very nearly so.

Finally, the writer wishes to express his appreciation to the discussers for their comments.

ANALYSIS OF FRAMES WITH NONLINEAR BEHAVIOR[a]

Discussion by Edward L. Wilson

EDWARD L. WILSON,[6]—The author has developed a method of analysis that realistically predicts the behavior of frame structures loaded into the plastic range. The method is simple to apply because it makes use of some of the well-known techniques of moment distribution. Since the deformation of the structure is fixed and the unknown forces are determined by an iteration procedure, problems of convergence are virtually eliminated. However, because structures are usually analyzed for a specific load rather than for a specific deformation, the method does have practical limitations.

However, an advantage of fixing the deformation pattern of the structure and then solving for the internal forces is that the secondary effects due to these displacements can be evaluated. If the structure that is analyzed by the author in Example A is also loaded with vertical forces as shown in Fig. 12, then the behavior of the structure becomes a function of the lateral displacement. This nonlinear behavior is due to the overall geometric change of the structure and is not caused by the nonlinear behavior of the individual structural elements. The secondary moments developed due to this displacement can be accounted for if the computation for load P is based on the geometry of the deformed structure. For this example, it is easily shown that the load in terms of internal moments and lateral displacement is given by the following equation:

$$P = \frac{M_A + M_B + M_C}{15 + 2\,k\,\Delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

The load P as a function of lateral displacement is shown by the dotted lines in Fig. 12. Detail analysis by the author's procedure was carried out for lateral displacements of one and two feet. For this structure the importance of considering the effects of vertical loads is clearly illustrated. It is apparent that the maximum load must be determined by a trial and error procedure.

The nondimensional moment-curvature relationship developed by the author is based on an uniaxial state of stress. In general this is not the stress condition in flexural members, and tests show that the actual moment-curvature relationship tends to be much smoother.[7] Another approach, which is perhaps

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[a] June, 1960, by Alfredo Hua-Sing Ang.

[6] Graduate Student, Dept. of Civ. Engrg., Univ. of California, Berkeley, Calif.

[7] "Commentary on Plastic Design in Steel: General Provision and Experimental Verification," Progress Report No. 2 of the Joint WRC-ASCE Committee on Plasticity Related to Design, Proceedings, ASCE, Vol. 85, No. EM 3, July, 1959.

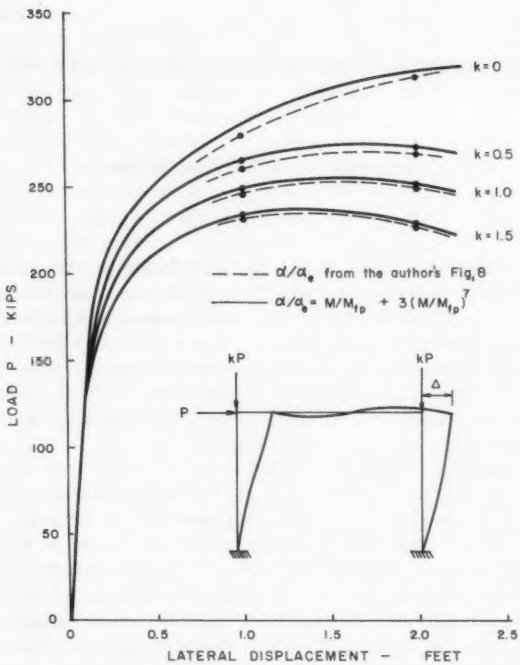

FIG. 12.—LOAD VS. DISPLACEMENT FOR PORTAL FRAME

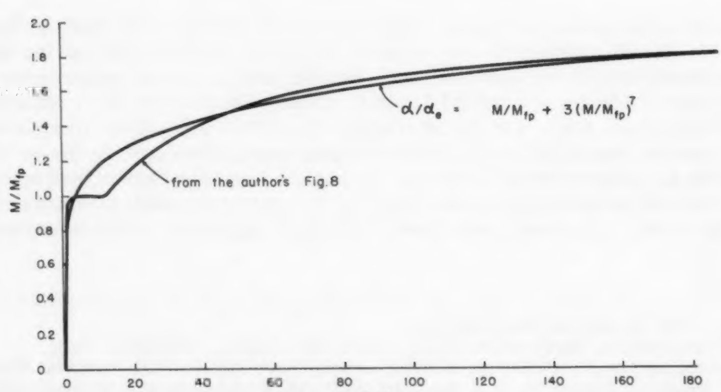

FIG. 13.—NONDIMENSIONAL MOMENT-CURVATURE RELATIONSHIP

equally valid, is to assume that the moment-curvature is a continuous function and can be expressed by the nondimensional equation

$$\alpha/\alpha_e = M/M_{fp} + \left(M/M_{fp}\right)^n \quad \ldots\ldots\ldots\ldots\ (12)$$

in which a, n, and M_{fp} can be determined from the material and cross sectional properties of the member. The form of this equation is similar to the well-known stress-strain equation developed by Ramberg and Osgood.[8] For values of a = 3 and n = 7, Fig. 13 illustrates the comparison of the two methods of representing the moment-curvature. The advantage of expressing the moment-curvature as a continuous function is that the end rotations can be expressed in terms of end moments by the equation

$$\phi_{ij} = \frac{L}{6\,E\,I}\left[2\,M_{ij} - M_{ji} + \frac{6a}{M_{fp}^{\,n-1}}\left(\frac{(n+1)\,M_{ij}^{\,n+2} + (n+2)\,M_{ji}\,M_{ij}^{\,n+1} - M_{ji}^{\,n+2}}{(n+2)\,(n+1)\left(M_{ij} + M_{ji}\right)^2}\right)\right] \ldots (13)$$

This equation is equivalent to the author's Figs. 3 and 4 which were obtained by a numerical integration procedure. Of course Eq. 13 is not restricted to a specific material or type of cross section.

A number of frame structures composed of members whose nonlinear behavior is represented by Eq. 13 have been analyzed.[9] This approach was developed under a National Science Foundation Research Grant. As part of this investigation an IBM 704 computer program was developed which solves the general nonlinear frame structure. To eliminate problems of convergence an incremental load method was used. This method approximates the analysis of a structure by the sum of the analyses of a series of structures, each acting linearly under a small increment of load. The displacements and member forces are obtained after each load increment. Therefore, the complete behavior is obtained as the structure is loaded.

As in the author's procedure, if the geometry of the structure is based on a specific deformed shape then the effects of these deformations can be evaluated. Because the structure is solved by a load increment method the solution will be correct when the displacements correspond to this deformed shape.

The author's Example A was analyzed by this approach. The solid lines in Fig. 12 show the results of a computer analysis of this structure. These results agree very well with those obtained by the author's method. This indicates that for this structure the difference caused by the approximation of the moment-curvature by a continuous function is small. The computer time required for this complete set of results was less than ten minutes.

The nonlinear characteristics of joints and foundations can also be considered with the same computer program if they are idealized as separate structural elements with nonlinear properties.

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8 "Description of Stress-Strain Curves by Three Parameters," by W. Ramberg and W. R. Osgood, NACA TN 902, July, 1943.

9 "Matrix Analysis of Non-Linear Structures," by E. L. Wilson, Proceedings, ASCE Second Conference on Electronic Computation, Pittsburgh, September, 1960.

VIBRATIONS AND STABILITY OF PLATES UNDER INITIAL STRESS[a]

Discussion by E. F. Masur

E. F. MASUR,[4] M. ASCE.—The authors' conclusion that beams or plates may become unstable as a result of shearing forces alone is an interesting contribution, especially since it is rather unexpected. Even in, as the authors point out, the beam would have to be uncommonly long for such a phenomenon to occur, the authors' development is nevertheless of basic interest and throws considerable light on the general question of elastic instability of plates and beams.

The authors have chosen to present their results in terms of Cartesian components. Undoubtedly, the most natural language for the problem discussed by the authors is that of general tensors. It may be surmised that the authors selected their own method of presentation in order to reach a larger group of readers. If so, the penalty has certainly been one of increased complexity and length, to say nothing of a measure of obscurity which is inherent in equations covering a page or more, as does the authors' Eq. 8. For this reason, and in view of what follows, permission is requested to conduct this discussion in the more compact language of Cartesian tensors.

The authors have developed their theory "from the bottom up," that is, without making use of more general formulations that have appeared in recent years. Actually, the problem discussed in the paper represents a special case of the general instability of elastic bodies. Substitution of a particular buckling mode into the equations governing the general case then leads to the special problem treated in the paper under discussion.

If the deformations prior to the onset of instability are neglected, as is done in the paper, it can readily be shown[5] (16), (17), (18) that the Lagrangian function takes the form

$$L = \frac{1}{2} \int_t \int_V \left(S_{ij}\, \bar{u}_{k,i}\, \bar{u}_{k,j} + \frac{\partial^2 \phi}{\partial \epsilon_{ij}\, \partial \epsilon_{kl}}\, \bar{u}_{i,j}\, \bar{u}_{k,l} - \rho\, \dot{\bar{u}}_i\, \dot{\bar{u}}_i \right) dV\, dt \quad ..(63)$$

In Eq. 63 S_{ij} refers to the already existing ("initial") system of stresses, while $\bar{u}_i$ stands for the displacement field which is superimposed on the body in equilibrium. Roman subscripts take on the values 1, 2, and 3, subscripts preceded by a comma denote differentiation with respect to the corresponding coordinate, and repeated subscripts denote summation over their range. Unless otherwise indicated the notation in this discussion follows as closely as possible that of the original paper. It may also be of interest to point out that the first two terms in the integrand in Eq. 63 are obtained by expanding the potential energy in a Taylor series near the condition of equilibrium and retaining only the quadratic terms. The last expression in the integrand is the kinetic energy.

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[a] June, 1960, by George Hermann and Anthony E. Aremenakas.

[4] Prof. of Engrg. Mech., The Univ. of Michigan, Ann Arbor, Mich.

[5] Numerals in parenthesis, thus (1), refer to corresponding items in the Bibliography.

For the sake of brevity the classical plate theory will be assumed here, although an extension to include the effect of shear deformations and rotatory inertia can be made without difficulty. In this spirit the displacements are assumed to be of the following type

$$\bar{u}_\alpha(x_1, x_2, z; t) = u_\alpha(x_1, x_2; t) - z\, w_{,\alpha} \quad (\alpha = 1,2) \quad \ldots\ldots\ldots \quad (64a)$$

$$\bar{u}_3 = w(x_1, x_2; t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (64b)$$

Eqs. 64 are the authors' Eqs. 3 with the approximation $\Psi_\alpha = -w_{,\alpha}$ already included. When now Eqs. 64 are substituted in Eq. 63 and when the form of the strain energy density ϕ is assumed to be that of an isotropic elastic material in the condition of plane stress, then the Lagrangian function L becomes

$$\begin{aligned} L = {} & \frac{D}{2} \int_t \int_A \left[(1-\nu) w_{,\alpha\beta}\, w_{,\alpha\beta} + \nabla\, w_{,\alpha\alpha}\, w_{,\beta\beta}\right] dA\, dt \\ & + \frac{E\hat{p}}{2} \int_t \int_A \left[\frac{1-\nabla}{2} \left(u_{\alpha,\beta}\, u_{\alpha,\beta} + u_{\alpha,\beta}\, u_{\beta,\alpha}\right) + \nabla\, u_{\alpha,\alpha}\, u_{\beta,\beta}\right] dA\, dt \\ & + \frac{1}{2} \int_t \int_A \left[N_{\alpha\beta}\left(u_{\gamma,\alpha}\, u_{\gamma,\beta} + w_{,\alpha}\, w_{,\beta}\right) - \left(M_{\alpha\beta}\, w_{,\gamma}\right)_{,\beta} u_{\gamma,\alpha} - m_\alpha\, w_{,\gamma}\, u_{\gamma,\alpha}\right] dA\, dt \\ & - \frac{h}{2} \int_t \int_A \rho\left(\dot{u}_\alpha\, \dot{u}_\alpha + \dot{w}^2\right) dA\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (65) \end{aligned}$$

In this expression, the first integral represents the usual energy in bending, the second the energy of the "membrane stresses," the third can be interpreted as the work done by the existing stress system, and the fourth is the kinetic energy as before. D and $E\hat{p}$ denote the same stiffness coefficients as in the paper. The terms that have been ignored in Eq. 65 are of the same type as those ignored by the authors. Moreover, the work done by the existing shear forces Q_α has been eliminated by means of the equations of equilibrium

$$M_{\alpha\beta,\beta} - Q_\alpha + m_\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (66)$$

in which m_α represents the kind of "body couples" envisaged by the authors. As mentioned before, terms corresponding to rotatory kinetic energy have been dropped.

From the functional in Eq. 65 the equations of motion can be derived in the usual manner through Hamilton's principle. This leads to

$$\left[N_{\beta\gamma}\, u_{\alpha,\beta} - \left(M_{\beta\gamma}\, w_{,\alpha}\right)_{,\beta} - m_\gamma\, w_{,\alpha} + E p\left(\frac{1-\nabla}{2}\, u_{\alpha,\gamma} + \frac{1+\nabla}{2}\, u_{\gamma,\alpha}\right)\right]_{,\gamma} = \rho\, h\, \ddot{u}_\alpha \quad \ldots (67a)$$

$$\left(N_{\alpha\gamma}\, w_{,\alpha} + M_{\alpha\delta}\, u_{\gamma,\alpha\delta} - m_\alpha\, u_{\gamma,\alpha}\right)_{,\gamma} - D\, w_{,\alpha\alpha\beta\beta} = \rho\, h\, \ddot{w} \quad \ldots (67b)$$

and to a set of natural boundary conditions that are not reproduced here for the sake of brevity. It may be remarked that Eq. 63, and hence all subsequent developments, are based on the assumption of a loading system which is constant and conservative; in other words, the authors' terms of the type Δm^i and m^a introduced in their Eq. 29 are seen to vanish, although they can be included in the discussion with little additional difficulty.

Eqs. 67 are practically identical with the authors' Eqs. 38 and 40 if expressed in Cartesian components. A slight discrepancy, however, does arise. For example, in the first of Eqs. 67 there arise terms of the type $\frac{\partial}{\partial x}\left(N^i_{xx} \frac{\partial u}{\partial x}\right)$

(using the authors' notation), which are not present in the original equations. These terms are usually of small significance but must be included for the sake of completeness.

The discrepancy arises out of the stress-strain law assumed in the original paper. The authors present several possibilities, among which they cite Eq. 23. This they reject on the basis that it would lead to the relation $\tau_{xx} = (1+E_x)(S_{xx} + t_{xx})$ "instead of the anticipated $\tau_{xx} = S_{xx} + t_{xx}$." As a result the authors resort to the artificial set of Eqs. 25.

The argument adopted in rejecting Eq. 23 is not tenable. The stress components τ_{ij} that the authors introduce can be shown to lack true tensorial properties since they are based on the instantaneous directions but on the original cross-sectional areas. It is therefore not clear why the relationship just quoted should be anticipated. The correct stress-strain relationship is obtained through the use of the Kirchhoff-Trefftz stress tensor, S_{ij}, which is related to the actual stress tensor by multiplying the latter by ρ_o/ρ, where ρ is the instantaneous mass density and ρ_o the original mass density. Because, as the authors themselves point out, $S_{ij} = \partial\phi/\partial\epsilon_{ij}$, it follows that

$$S_{ij} + \Delta S_{ij} = \frac{\partial\phi}{\partial\epsilon_{ij}} + \frac{\partial^2\phi}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\,\Delta\epsilon_{kl} + \ldots \qquad (68a)$$

or

$$\Delta S_{ij} = t_{ij} = \frac{\partial^2\phi}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\,\Delta\epsilon_{kl} \qquad (68b)$$

In other words, the authors' t_{ij} represents the change in the value of the Kirchhoff-Trefftz tensor. The right side of Eq. 68 is obtained through a Taylor expansion and linearization; the second derivative is evaluated at the appropriate value of ϵ. If, moreover, ϕ is assumed to be quadratic in the strains, then the second derivatives appearing in Eq. 68 are constants and the additional Kirchhoff-Trefftz stress components are related to the additional strains through the usual linear (Hooke's Law) coefficients.

Because the form of ϕ is not specified, the temptation is great to assume that the authors' Eq. 25 can, nevertheless, be obtained through a suitable choice of the energy function. It will now be shown that this is not possible. In fact, in view of the vanishing of the stresses for vanishing strains (thus eliminating the linear terms), the most general form of ϕ, except for an irrelevant constant, is

$$\phi = \frac{1}{2}\,c_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl} + \frac{1}{6}\,d_{ijklpq}\,\epsilon_{ij}\,\epsilon_{kl}\,\epsilon_{pq} + \ldots$$

$$c_{ijkl} = c_{klij} = \ldots \quad d_{ijklpq} = d_{ijpqkl} = \ldots \qquad (69)$$

in which the indicated symmetry of the coefficients can be deduced by the usual argument. The constants c_{ijkl} represent the customary approximation in linear elasticity.

It follows from Eqs. 69 and 68 that the stresses S_{ij} and their increments ΔS_{ij} are governed by

$$S_{ij} = \frac{\partial\phi}{\partial\epsilon_{ij}} = c_{ijkl}\,\epsilon_{kl} + \ldots \qquad (70a)$$

$$\Delta S_{ij} = \left(c_{ijkl} + d_{ijpqkl}\,\epsilon_{pq}\right)\Delta\epsilon_{kl} + \ldots \qquad (70b)$$

In particular,

$$\Delta S_{11} = \left(c_{11kl} + d_{11pqkl}\epsilon_{pq}\right)\Delta\epsilon_{kl} + \ldots \qquad (71)$$

On the other hand, when the nature of the coefficients c_{ijkl} as representing Hooke's Law is remembered, the authors' Eq. 25 implies, inter alia, that

$$\Delta S_{11} = c_{11kl}\,\Delta\epsilon_{kl} - S_{11}\,\Delta\epsilon_{11} + \ldots$$

$$= c_{11kl}\,\Delta\epsilon_{kl} - c_{11pq}\,\epsilon_{pq}\,\Delta\epsilon_{11} + \ldots \qquad (72)$$

in which the second equality is obtained by substituting the first of Eqs. 70 with $i = j = 1$.

The two formulations, Eqs. 71 and 72, must be identical for all values of ϵ_{pq}. By comparison it follows that

$$d_{11pq11} = -c_{11pq} \qquad (73a)$$

$$d_{11klpq} = 0 \quad (k \neq 1 \text{ or } l \neq 1) \qquad (73b)$$

where the symmetry properties of the coefficients have been utilized in Eq. 73b.

It can easily be demonstrated that Eqs. 73a and 73b are mutually incompatible. For example, the former implies that $d_{112211} = -c_{1122}$, whereas the latter postulates the vanishing of the same coefficient. Since c_{1122} is, in general, different from zero (and, in fact, equals Lamé's constant λ for an isotropic material), the contradiction has been established. In other words, it is not possible to select an energy function ϕ which will satisfy both the authors' Eq. 25 and, simultaneously, Eq. 68, which must be considered basic because it is derived from thermodynamic principles alone. Similar considerations make it necessary to reject the authors' Eq. 24.

As the final point in this discussion the validity of ignoring the pre-buckling deformations will be analyzed. The authors point out correctly that the error thus introduced is negligible if the pre-buckling stresses are small in comparison with the elastic constants—that is, if the pre-buckling strains are small as compared with unity. Actually, this is a necessary but not sufficient condition. It can readily be shown, and is, in fact, apparent from study of the references cited here, that the simplification introduced by the authors is valid only if all the initial displacement gradients are small. This means that both initial strain as well as rotation components must be negligible. Examples in which the latter may be significant abound in the literature. For example, the instability (that is, snapping) of a shallow spherical cap cannot be analyzed properly without including the effect of initial deformations. In this case the strains are small while the rotations are not.

Acknowledgment.—This discussion is connected with some of the work being carried out under National Science Foundation sponsorship.

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ULTIMATE STRENGTH OF OVER-REINFORCED BEAMS[a]

Discussion by Iqbal Ali

IQBAL ALI,[13] M. ASCE.—The authors are to be congratulated for their excellent paper on the ultimate strength of over-reinforced beams. It is a significant contribution and constitutes another step towards solution of a more general problem. This problem is concerned with estimating the ultimate strength of concrete members, subject to any given mode of loading, by identifying the same with the maximum internal resistance that can be mobilized, in accordance with the stress-strain response of the concrete, rather than on the basis of more or less arbitrary criteria for its actual rupture.

The writer had, some time ago, presented a similar analysis[14] of the ultimate flexural strength of both over and under-reinforced beams, using as a basis the stress-strain curve of concrete in axial compression. An idealized relation of an exponential form, as suggested by Smith and Young,[15] was used for demonstrating the principle. The general quadratic relation proposed by the authors, is a distinct improvement, in view of its greater flexibility in defining widely varying curve forms.

It is felt, that it would, perhaps, be relatively simpler to obtain the necessary stress-strain curves for concrete in compression, directly from axially loaded specimens, instead of using beam tests. It would, of course, be necessary to employ some suitable device to eliminate feed back from the testing machine, to enable the descending portion of the curve to be traced. Determination of such curves for concrete made with different types of aggregates, and for different rates of loading, would provide valuable additional data.

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[a] June, 1960, by Ladislav B. Kriz and Seng-Lip Lee.

[13] Research Officer, Andhra Pradesh Engrg. Research Dept., Hyderabad, India.

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EXPERIMENTAL STUDY OF BEAMS ON ELASTIC FOUNDATIONS[a]

Discussion by L. F. Stephens

L. F. STEPHENS.[8]—The author is to be commended for using a model to solve the problem of a beam on an elastic foundation, and especially for dealing with the difficult case in which the supporting medium possesses non-linear load-deflection characteristics. However, the type of model used, while suitable for an engineering laboratory, would hardly be suitable for use in a design office.

In the writer's opinion model analysis is a most useful aid to the designer, as an alternative to numerical analysis or as a means of providing a quick independent check on long and tedious computations. But such models should be capable of being quickly and cheaply made, and, in fact, the most useful structural models for the ordinary designer are those that can be manipulated on the drawing-board.

The writer has developed a "large-displacement" technique for solving the problem of the elastically supported beam. The model beam consists of a perspex spline of constant thickness, the depth of the spline being varied to correspond with variations in the flexural rigidity of the prototype beam. The elastic supports, which correspond with the coil-springs in the author's model, consist of cantilever beams cut from the same sheet of perspex. The depths and lengths of the cantilevers are arranged to reproduce any required stiffness of the supporting medium. A foundation material of the Winkler type, with a constant reaction modulus, is thus represented by a series of cantilever beams of constant length and depth. The cantilever beams are connected to the main beam by pin-jointed connecting pieces or struts, the whole assembly being mounted flat on the drawing-board. The cantilevers are attached to the board by two or three tightly fitting panel pins. To reduce friction the board may be put into the near-vertical position. The method of operation of the model is as follows

The first connecting strut at the left-hand end is removed and replaced by a special strut that has a series of holes drilled through it along its length. Using this special strut the connection between the first cantilever support and the beam is lengthened, thus imparting an upward deflection to the main beam and a downward deflection to the first cantilever. The curved form of the main beam is traced on the drawing paper with a fine pencil, noting particularly the positions at each of the elastic supports and at the location of any concentrated loads on the prototype beam. The downward deflection of the first cantilever is also noted. The special strut is then removed and replaced, now using a pair

[a] June, 1960, by Robert L. Thoms.

[8] Lecturer, Civ. Engrg., University College, Dublin, Ireland.

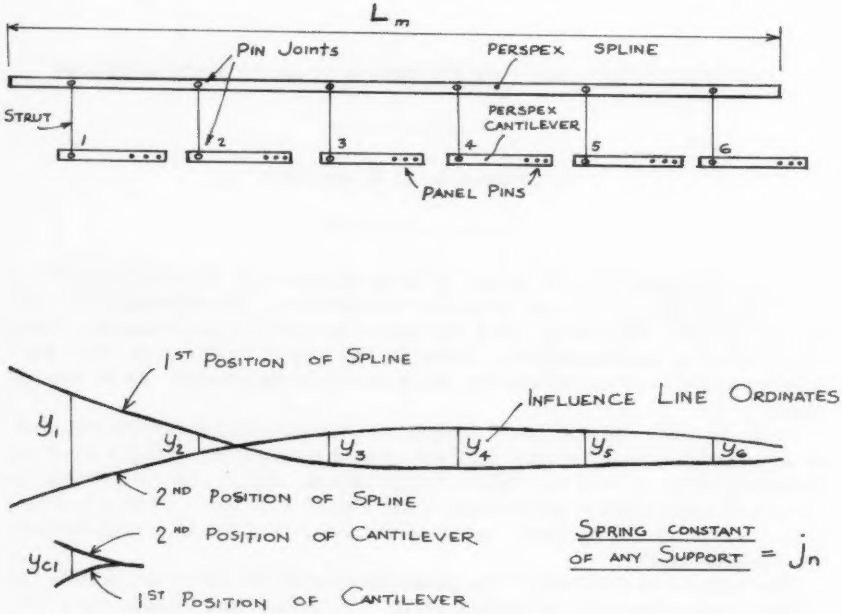

FIG. 7.—TOTAL DISPLACEMENT OF 1ST SUPPORT = $(y_1 + y_{c1}) = Y_1$

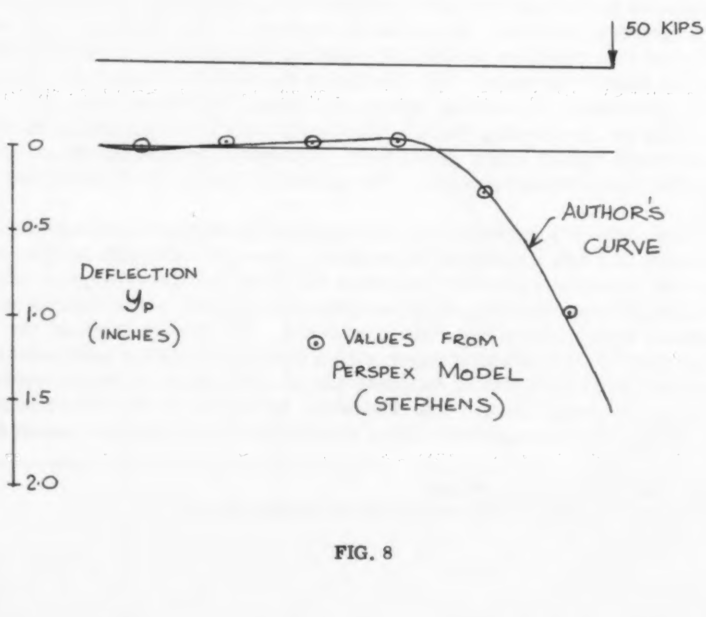

FIG. 8

of holes that are closer together, which will cause the main beam to deflect downwards and the cantilever to deflect upwards, and again the positions are traced on the drawing paper.

The total displacement that has thus been applied to the connecting strut is the sum of the main beam and cantilever displacements at position 1 $\left(y_1 + y_{c1}\right)$. Let this total displacement be Y_1. The ordinates y measured between the two extreme positions of the main beam, divided by Y_1 are then, by Muller-Breslau's principle, an influence line for the thrust in the first supporting strut. Influence lines for each elastic support are obtained in this way, and, hence, the elastic reactions may be found for any system of loading. As the method is an indirect method based on Muller-Breslau's principle, any system of loading may be considered once the influence lines have been obtained, and it is not necessary, as in the author's set-up, to take a new set of readings for every loading case. Distributed loads do not present any particular problem.

Due to minor variations in the stiffness of the supporting cantilevers, the accuracy obtained by the above procedure may be rather disappointing, but greatly improved accuracy may be obtained by the following considerations. If, in the prototype beam, there were a concentrated load acting on the beam

TABLE 3

Location $(X)_p$ in inches	Deflection $(Y)_p$, in inches.	
	Author	Stephens
50	-1.00	-0.97
150	-0.24	-0.27
250	+0.06	+0.05
350	+0.04	+0.05
450	0.00	+0.02
550	0.00	0

at each elastic support, the value of this concentrated load being numerically equal to the stiffness of the adjacent spring, then the beam would deflect a distance of 1 in., uniformly throughout its length, and the load on any spring would be numerically equal to its stiffness. Now consider the influence line obtained for support No. 1 (See Fig. 7). To satisfy the previous requirement we should have

$$\sum j_n y_n = j_1 Y_1 \qquad (13)$$

in which j_n is the spring constant of any support.

In practice it will be found that this equation is not exactly satisfied. We may, however, use the valve $\frac{\sum j_n y_n}{j_1}$ instead of the value Y_1 when obtaining the scale of the influence line, and this will give much better results. The displaced positions of the cantilever beams need not be recorded at all, but in fact it is better to record these and to carry out the above check to prevent gross errors.

Using a large number of supports in the model, the stiffness of each cantilever is made equal to the foundation stiffness per unit length, multiplied by

the spacing of the supports. If a small number of supports is used, it is better to take the stiffness corresponding to the reaction in a continuous beam on rigid supports.

The method may also be used to solve cases of the type described in the paper, where the load deflection curves for the foundation are non-linear. When the first set of pressures has been determined using approximate initial values of the secant soil moduli, and the deflections determined, a second more accurate set of values for the effective secant soil moduli may be found, and the lengths of the cantilever supporting beams adjusted accordingly.

The example of the laterally loaded pile solved by the author has been analyzed in this way, using a spline 30 in. long by 1/16th in. thick, and using only six elastic supports. Only two cycles of the iteration process were used, because the scales of the soil reaction curves were not shown in Fig. 3 of the paper. The writer had to reproduce these from the information given in Table 2 of the paper and by sketching in the curves approximately by inspection of Fig. 3, but obviously this is not a satisfactory procedure.

Nevertheless, the agreement with the author's predicted deflection curve is remarkable. See Fig. 8 and Table 3.

This method has been used by the writer to solve many problems of elastically supported beams.

STRESSES DUE TO THERMAL GRADIENTS IN REACTOR SHIELDINGS[a]

Discussion by O. C. Zienkiewicz

O. C. ZIENKIEWICZ,[10] F. ASCE.—The case discussed by the authors is typical of the many complex thermo-elastic problems presented in the design of reactor shields. As the assumption of independence of thermal and elastic properties is introduced the investigation can be subdivided in the usual manner into two distinct phases dealing with the temperature and stress distribution aspects separately. It is in connection with the second phase that the writer's comments are concerned. In problems of structural analysis numerous approaches are possible to even the most standard types of examples and it appears from a literature survey that in the field of thermal stress analysis of simple structures the picture is similar. The approach suggested by B. Boley and used ably in the paper under discussion is a typical method of analysis which, however, in more complex conditions becomes difficult to follow. An alternative method of approach that the writer will refer to as the "method of restraint" has in his experience proved extremely valuable in dealing with numerous thermal stress problems.

The "method of restraint," valid for all linear elastic structures consists of three distinct stages: Stage I - computation of stresses throughout the structure due to thermal changes alone, with the assumption that all elementary displacements are prevented; Stage II - computation of external forces necessary to maintain the structure in the undeformed position; State III - imposition of forces opposing those computed at Stage II on an unstressed structure. Clearly the final state of stress results from a superposition of stresses computed at Stage I and Stage III.

The advantages of this approach are considerable. In the first place the computation of stresses at Stage I is almost trivial, resulting in compressive stresses equal to $E \alpha T$ for the cases in which Poissons ration of zero is assumed. In addition the computations involved at Stage III for a structure under external loads are of a type with which the engineer is familiar and for which many known solutions are available. Often the solution to this part can be derived by simple inspection.

The computation of Stage II involve only the concepts of statics and present little difficulty.

To illustrate the above remarks let an arch similar to the one discussed in the paper be considered but with a more general type of temperature distribution $T(x,y,t)$ which now may be also a function of axial (x) as well as radial (y) position.

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[a] June, 1960, by M. L. Baron and M. G. Salvadori.

[10] Prof. Civ. Engrg., Northwestern Univ., Evanston, Ill.

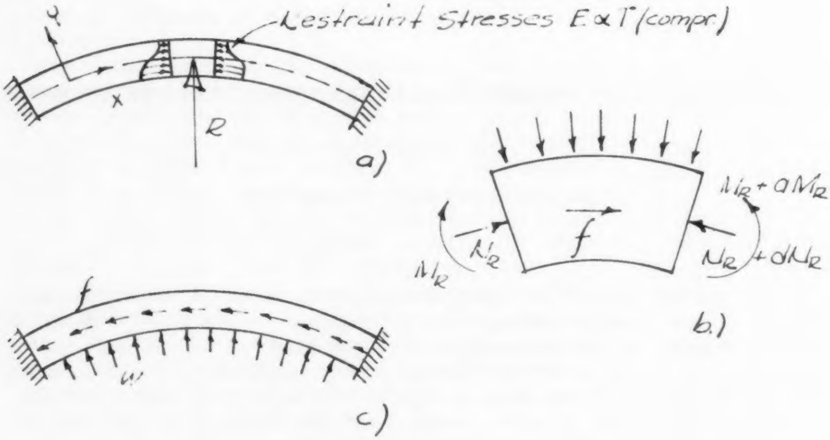

FIG. 6.—TYPICAL STAGES IN THE SOLUTION OF THERMAL STRESSES IN AN ARCH SUBJECT TO AN ARBITRARY TEMPERATURE CHANGE

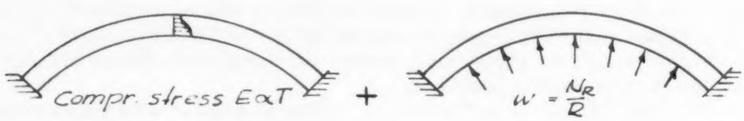

FIG. 7.—SOLUTION OF THE THERMAL STRESS PROBLEM IN AN ARCH SUBJECT TO A RADIAL TEMPERATURE CHANGE

Fig. 6(a) shows the restraining stresses, E α T, developed if no displacements are permitted. These result at any section in a normal force

$$N_r = \int E \alpha T \, dA \qquad (34)$$

and moment

$$M_r = \int E \alpha T y \, dA \qquad (35)$$

From Fig. 6(b) it is evident that to maintain equilibrium it is necessary to apply a radial load

$$w = \frac{N_r}{R} - \frac{d^2 M_r}{dx^2} \qquad (36)$$

and a tangential force

$$f = - \frac{dN_r}{dx} \qquad (37)$$

to each element of unit length. Lastly in Fig. 6(c) the problem of Stage III is given which can be solved by any standard procedure.

In the particular case discussed in the paper both N_R and M_R are invariant with x and therefore the problem of stress distribution is particularly simple requiring only a solution of an arch subject to a uniform outwards radial load

$$w = \frac{N_r}{R} \quad \text{. . . . .} \quad (38)$$

and addition to the stresses resulting from the preceding compressive stresses $E \alpha T$ as shown in Fig. 7. The solution for an uniformly loaded segmental arch is available in standard texts and together with the restraint stresses will give the final stress distributions identical to those given in the paper.

In Fig. 5 the stresses resulting due to a particular temperature distribution are shown. It is not indicated to which section of the arch these stresses are applicable. There is obviously a considerable variation in stresses from the crown to the abutments in the arch which depend on its thickness to radius ratio.

The authors discuss the use of the Duhamel integral in the evaluation of temperature distribution from known solutions to a step imput of temperature. This approach is very valuable in the superposition of linear effects and as an extension of this one may well consider the use of a diagram similar to that of Fig. 5 an "influence chart" for a direct computation of stresses due to an arbitrary temperature pulse. This would eliminate the repetition of the solutions of the purely structural problem for different temperature distributions.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Department of Conditions of Practice are identified by the symbols (PP). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper number are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper [illegible] is identified as [illegible] which indicates that the paper is contained in the [illegible] issue of the Journal of the [illegible] Division during [illegible].

VOLUME 85 [illegible]

DECEMBER: [illegible]

VOLUME 86 (1960)

JANUARY: [illegible]

FEBRUARY: [illegible]

MARCH: [illegible]

APRIL: [illegible]

MAY: [illegible]

JUNE: [illegible]

JULY: [illegible]

AUGUST: [illegible]

SEPTEMBER: [illegible]

OCTOBER: [illegible]

NOVEMBER: [illegible]

DECEMBER: [illegible]

c. Discussion of several papers, grouped by Divisions.

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PART 2

DECEMBER 1960 — 45
VOLUME 86

NO. EM6
PART 2

Your attention is invited

NEWS
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DIVISION ACTIVITIES
ENGINEERING MECHANICS DIVISION
Proceedings of the American Society of Civil Engineers

NEWS

December, 1960

CHAIRMAN'S ANNOUNCEMENTS

Professor E. P. Popov, incoming chairman of the Executive Committee, Engineering Mechanics Division of ASCE, and Professor of Structural Mechanics, University of California, Berkeley, suggested the following items of possible interest to members of the EMD.

1. The ASCE has just established the Von Karman Medal, and the first medalist, William Prager, received the award in Boston during the annual meeting. Dr. Theodore Van Karman, in whose honor the Medal was established, was also at the meeting and he was presented with a replica of the medal as a memento.

2. The status of two manuals being prepared by the EMD committee is as follows: the "Commentary on Plastic Design" passed all the hurdles and shortly will be published as a joint ASCE-WRC manual. A second manual on "Design of Structures to Resist Nuclear-Weapons Effects" has been available in report form for several weeks and was reported on at the recent annual meeting in Boston. The deadline for comments by reviewers was December 15, 1960. The EMD has strongly recommended that the publication of this work proceed with a minimum of delays.

3. Two publications of interest to persons working in mechanics are now available:

a. Proceedings of the IUTAM Symposium on the Theory of Thin Elastic Shells, held in Delft, August 1959, has been published by North Holland Publishing Company and is available through the Interstate Publishers, Inc., in New York.

b. Proceedings of the First Symposium on Naval Structural Mechanics, held at Stanford University, August 1958, is now available from Pergamon Press, edited by J. N. Goodier and N. J. Hoff, 1960.

4. The International Association for Shell Structures is planning additional symposia on the subject of shells. The first colloquim is being scheduled in cooperation with the International Union of Testing and Research Laboratories for Materials and Structures. This colloquim will deal with "shell research" and will be held in Delft, Netherlands, from August 30 through September 2, 1961. The second colloquim will take place in Brussels, Belgium, from

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September 4-6, 1961. It will deal with "simplified calculation methods."

EXCERPTS FROM ANNUAL REPORT

Oct. 1, 1959 to Sept. 30, 1960

Executive Committee

During this past year the Engineering Mechanics Division has, as usual, continued its effort to improve Society publications and policy as well as to carry on an expanding basis of operations so characteristic of any active Technical Division. Some of this activity is described in the reports of the individual Division Committees which follow this introductory summary. These also include the current committee membership lists.

Two manuals sponsored by the Division are nearing the final stages. These include the "Commentary on Plastic Design in Steel" on which discussion closed September 1, 1960, and the "Design of Structures to Resist Nuclear-Weapons Effects" which is still open for discussion.

The Newsletter continues its active function under the editorship of Donald L. Dean, now of the University of Delaware. The volume of papers being processed by the Division has doubled in the past two years and now averages about sixty-five per year. About one-third are rejected or withdrawn. This increase required issuing the EMD Journal on a bi-monthly basis beginning in April, rather than quarterly as heretofore.

Previous Division recommendations on the Society Publication Policy were accepted. Under the new procedure all papers cleared after December 1, 1959 and published in the Journals will be re-issued in the Transactions, collated, and indexed. However, some final adjustment may be needed in this matter since the volume of printed material must be controlled by some budgetary limitations. It would appear that either a drive to have all Divisions consider shortening the maximum allowable length of papers from the present 18000 word limit or a raising of acceptance standards for papers would now be in order. The requirement that only Journal papers which attracted constructive discussion be published in the Transactions was withdrawn after a number of Divisions, including EMD, objected.

The increase in papers being submitted for presentation to EMD will require their being read at regional as well as annual meetings in the future. At the 1960 annual meeting EMD scheduled more sessions than any other division—8 in all at which 29 papers will be presented. In addition EMD again cosponsored the West Coast Conference on Applied Mechanics with the ASME on June 27-29 at California Institute of Technology and supplied 3 papers for this program. EMD also scheduled the first simultaneous conference between a technical division and a meeting of State ASCE student chapters. This meeting took place at Purdue University on May 5-6. Eleven papers were presented. Dr. Mario Salvadori, a former EMD Committee Chairman, gave the principal banquet address to this student and structural mechanics conference. Attendance varied between 85 to 100.

Dr. Bruce Johnston and Mr. Elmer K. Timby (contact member) represented EMD at the ASCE Technical Procedure Conference in Omaha on April 29. They presented EMD recommendations to the conference and a series of suggestions were prepared for the Division Activities Committee.

The Fluid Dynamics Committee organized a new task committee on "Flow Past Objects Immersed in Boundary Layers" and is studying the desirability of incorporating oceanography and meteorology in its sphere of activity.

The Von Karman Award became an official Society prize when the special committee headed by Dr. Arthur T. Ippen finished its fund raising and selected a suitable medal. The first recipient was nominated by the EMD Advisory Board under Dean John McNoun's direction. A telephone conference call system scheduled at a predetermined time was used for final discussions of the several nominees. This proved to be quite successful and considerably less expensive, financially and time-wise, than a Board meeting would have been.

The EMD Executive Committee met in Washington on October 20, 1959 and again in Lafayette, Indiana on May 7, 1960. As usual the agenda were full and the discussion lively on most topics.

One of the several men nominated in May 1958 by EMD for the Society Research Award received this prize in 1959. A like number was nominated again this year.

EMD membership continues to gain at a faster rate than division membership generally. A 13.3% growth was exhibited this past year and the division now ranks 7th in size. No report such as this, prepared by a retiring chairman, would be complete without an expression of sincere gratitude for the magnificent cooperation extended by all divisional committee members. Our progress is a direct result of their faithful hard work and of the direction supplied by technical committee chairmen, other members of the Executive Committee, and especially the Secretary. The latter also would ask your chairman to thank all those who assisted them in turn.

Respectfully submitted,

D. H. Pletta, Chairman
Executive Committee

Committee on Elasticity

The activities of the committee consisted essentially of reviewing papers for publication and possible presentation at meetings sponsored by the Division.

A total of thirteen papers were received in the current year. Twelve of these papers and two more from the previous year were reviewed and returned to the Division Secretary.

At the Annual Meeting October 1959 two sessions on light weight construction were co-sponsored with the Structural Divisions. Further, the committee has contributed a paper to the regional meeting at Purdue University, and two papers to the ASCE-ASME West Coast Meeting at Pasadena.

Committee on Experimental Analysis and Analogues

Since October the committee has reviewed three manuscripts of which two were accepted and one rejected. In addition one session comprised of four new papers has been scheduled for the Boston meeting in October.

Committee on Fluid Dynamics

During the past year the Fluid Dynamics Committee has been chiefly concerned with increasing interest and activity of the Society in Fluid Mechanics. To help in achieving this the Committee for the first time is sponsoring two technical sessions at the Annual Convention to be held in Boston and has organized a new task Committee—"Flow Around Objects Immersed in a Boundary Layer."

The Task Committee on Mechanics of Stratified Flow is sponsoring one session with three outstanding papers scheduled for presentation. A second session on Computational Methods in Fluid Mechanics has been organized by Dr. J. W. Delleur and Dr. W. D. Baines.

The Task Committee on the Mechanics of Stratified Flow is making preparations for issuing a symposium publication. About a dozen papers have been presented at various ASCE meetings under sponsorship of the Task Committee. These papers will form the symposium publication together with a bibliography of the subject which has been prepared.

A new Task Committee on Flow Around Objects Immersed in a Boundary Layer has been organized. The Task Committee has the goal of bringing together knowledge from fields such as meteorology, oceanography and fluid mechanics to stimulate research on civil engineering problems of wind forces on structures, atmospheric pollution and sediment transport.

Four technical papers have been reviewed or are now in the process of review for publication in the EMD Journal. In addition, seven papers were prepared for presentation at the Annual Convention in Boston. Continuous effort by the Committee is being directed toward the stimulation of writing and submitting good papers on fluid mechanics to the EMD Journal even through a 100 percent increase in volume has been achieved over the previous years.

Committee on Mathematical Methods

The past year saw a decided spurt in the number of papers reviewed by the committee. In all a total of 13 papers have been reviewed or are under review at the present time. Only one paper has been accepted without major revision but three are currently under review. The rate of rejection was much higher than that for the Division as a whole. It was suggested that this was not due to the severity of the reviewers, but rather to the low caliber of most of the papers. One suggestion is that the area of activity in the Mathematical Methods Committee is less clearly defined than that of some of the other committees.

Committee on Mechanical Properties of Materials

The principal activity of the Committee on Mechanical Properties of Materials has been the arrangement of a one-day symposium held October 13 at the Annual Meeting of the ASCE in Boston. Several papers were presented in the two sessions.

Committee on Plasticity Related to Design

The Committee's activities this year have been concerned primarily with the "Commentary on Plastic Design in Steel," the routine processing of technical papers, and participation in programs of technical conferences. The committee thus far this year has received six new papers for review, plus several corrected and rewritten papers for a second review. The committee supplied several papers for presentation at the West Coast Conference. Two papers processed by the committee were presented at the May Conference on Structural Mechanics held at Purdue University.

Committee on Structural Dynamics

The activities of the Committee on Structural Dynamics have been concentrated on three major items: (1) Review of papers submitted for publication; (2) Preparation of a draft of a potential manual of engineering practice entitled "Design of Structures to Resist Nuclear Weapons Effects"; and (3) Planning of a program for two sessions at the Annual Meeting of ASCE in October 1960. The activity in each of these areas is summarized below.

At the time of the last annual report, four manuscripts were in the review stage. Of these four, two were accepted to publication and two were rejected. Since the last annual report, nine papers have been submitted for review. Of these, three were recommended for publication, two were recommended for publication after revision by the author, two were rejected, and two are still in process of review.

The draft of "Design of Structures to Resist Nuclear Weapons Effects" was completed by the Manual Subcommittee, Cmdr. H. L. Murphy, Chrmn. It is now being reviewed by members of the entire Committee on Structural Dynamics and a number of other interested people. Efforts have been initiated to have this manual published as a Manual of Engineering Practice by the Society.

Plans were made by this committee for two sessions at the Annual Meeting of ASCE in October 1960. These two sessions will have as their theme "Protective Construction" and will be co-sponsored with the Structural Division.

SECRETARY'S REPORT

Engineering Mechanics Division
American Society of Civil Engineers

As of October 7, 1960, a total of 200 manuscripts have been received for review since formation of the Engineering Mechanics Division in 1955. A breakdown of action concerning these 200 pages is noted below:

Total Manuscripts received	200	Rejected	44
Published	107	Withdrawn by Author	11
Accepted & Awaiting publication	16	Under Review	22

Of the 22 papers listed as "Under Review," six has been returned to the author for consideration of the reviewers' suggestions and recommendations. The remaining 16 are now being processed, five of which have been in the

reviewers' hands since June. Two of the papers currently in the review stage are scheduled for presentation at the Boston meeting.

Since October 1, 1959, a total of 62 papers have been received, 26 published, two withdrawn by author, 21 rejected, 12 approved and awaiting publication and 21 papers still under review.

The increase in rate of submissions is shown in the following comparison:

Oct. 1, 1955 - Sept. 30, 1956	27 papers
Oct. 1, 1956 - Sept. 30, 1957	21 papers
Oct. 1, 1957 - Sept. 30, 1958	36 papers
Oct. 1, 1958 - Sept. 30, 1959	54 papers
Oct. 1, 1959 - Oct. 7, 1960	62 papers
Total	200

A breakdown of papers by subject matter follows:

Committee	Papers received Since Oct. 1955	Papers received Since Oct. 1, '59
Elasticity	44	15
Experimental Analysis & Analogues	17	9
Fluid Dynamics	15	3
Mathematical Methods	29	11
Mechanical Properties of Materials	15	1
Plasticity	40	14
Structural Dynamics	29	8
West Coast Committee	11	1

Since October 1959, the rate of publication of manuscripts in the Journal of the Engineering Mechanics Division is suggested by the following comparison:

October 1959	9	June 1960	10
January 1960	7	August 1960	9
April 1960	7		

Papers received since July 1958 which have not been presented at any meetings through Boston total 47. Of these 20 have already been published, 7 are approved but not published and 20 are under review. It should also be noted, incidentally, that a number of papers scheduled for the 1960 Boston meeting have not yet been submitted to the Division for review.

Processing of manuscripts, with a few minor exceptions, is moving smoothly. One question regarding policy concerns the interval we should allow between the return of a manuscript to an author for revision and the date it is considered automatically withdrawn in lieu of author reply.

Respectfully submitted,

Edward Wenk, Jr.
Secretary

TECHNICAL NOTES

Dan Pletta, past chairman of the Executive Committee, has asked that the EMD members be reminded of the fact that short technical notes are welcome for publication in the Division Journal. Several other societies have a long

history of success in publishing technical notes and it is believed that they would serve a useful purpose for EMD members. They will be treated in exactly the same way as regular papers and hence will have rather speedy consideration and publication opportunities. These short technical notes will not necessarily be used for program material.

NEW COLUMN BOOK

Column Research Council has announced the publication of it's "Guide to Design Criteria for Metal Compression Members." This book of 112 pages presents a condensed summary of design criteria based upon recent as well as past research on metal compression members in buildings and bridges. The book is written especially for the engineer who either meets special problems not covered by standard specifications or is himself engaged in the preparation of specifications for such structures. Areas of interest covered include centrally loaded columns, compression member details, laterally unsupported beams, and beam-columns. Constructional metals under ASTM Specifications are covered, including the new ASTM A36 Structural Carbon Steel.

The price for this 8 1/2" x 11" leatherette bound volume is $5.00 per copy and orders may be placed with the

Secretary
Column Research Council
313 West Engineering Building
University of Michigan
Ann Arbor, Michigan

MINUTES OF COMMITTEE MEETINGS

West Coast Committee, June 27, 1960
CIT, Pasadena

The following items of interest were discussed jointly with the ASME Committee:

1. The West Coast Committee of the ASME Applied Mechanics Division is now to function essentially as a technical Committee for reviewing. The committee will select reviewers from an approved list of reviewers on the West Coast and reviews will be submitted directly to the West Coast Committee, rather than to JAM, to speed up the reviewing process.

2. Future meeting dates:

<u>1961</u> - University of Washington, Seattle, August 28, 29, and 30. (It was noted that the Midwest Conference on Solid Mechanics is scheduled on September 7, 8, 9, 1961).

<u>1962</u> - 4th U. S. National Congress at Berkeley in June - no West Coast Conference.

<u>1963-64</u> - Meeting date for 1963 tentatively set for week preceding Labor Day - meeting place open. The ASME Applied Mechanics Division Executive Committee has been asked to try to re-schedule the 1963 National Conference for Boulder to Boulder in 1964. In which case the 1964 West Coast Conference would be amalgamated with the National Conference at Boulder for that year.

A joint motion was passed to the effect that these committees desired that on every fourth year the West Coast Conference and the National Applied Mechanics Conference would be amalgamated and the meeting held in the west. These would be in the years of the International Congress (1964 would be the first year for this amalgamated conference). It was indicated that in these years the meetings could be extended to more than three days to accommodate the additional papers.

The possibility of having the meetings in hotels or resort areas was discussed, but dismissed due to cost and difficulty of transportation.

3. The ASME Committee requested that the ASCE Committee make an effort to have reprints of papers available at the conferences.

4. The ASME Committee indicated that the ASME was willing to continue to bear the cost of printing all of the programs for both ASME and ASCE use. However, it was indicated that the ASCE may be able to assist financially in other ways, such as in the cost of making special mailing lists from the lists of registrants.

Items from minutes of private ASCE committee meetings:

Committee Objectives - It was decided that the committee should attempt (i) To promote work in this subject in our own organizations and to demonstrate the need to national agencies and other organizations. (ii) To support proposals made for funds for the carrying through of this work. (iii) To arrange programs at ASCE meetings. (iv) To collect a symposium of related papers as published in ASCE Proceedings.

Task Committee on

"Flow Around Objects Immersed in a Boundary Layer"

September 8, 1960 Ann Arbor, Michigan

W. D. Baines reported on an air tunnel study of the pressure distribution on vertical walls and linear vertical profile. Due to limitations of the tunnel and instrument only qualitative results have been obtained.

For walls perpendicular to the flow the pressure gradients on the front face are not so steep as the stagnation pressure gradient of the flow. The narrower the building, the steeper is the surface pressure gradient.

For walls at an angle to the flow the surface pressures follow the same variation as the stagnation pressures.

In the wake downstream of the wall the pressure is constant at a value approaching that expected for the highest velocity in the mean flow if the flow had a constant profile.

Purpose of these studies is the defining of rules for analysis of wind forces on buildings with a natural wind velocity gradient.

Other staff members at Toronto are studying the diffusion of gases from industrial stacks. They have found that velocity gradients near the ground influence the plume shape.

J. E. Cermak reported on a study made of the separation zone behind a two-dimensional fence extending vertically into a boundary layer. The length of the zone was found to be proportional to the height of the fence and the variation was linear over most of the range.

Currently, investigations are being made of the diffusion from a point source in a boundary layer. It is planned to heat the floor in future work to

produce a temperature gradient across the layer and eventually to introduce two-dimensional topography downstream of the point of introduction of the tracer. It is hoped that from this study that scaling factors can be determined for model studies of atmosphere pollution besides assisting in theoretical developments.

It is proposed that studies be also made of diffusion in the presence of a rough surface. Roughnesses elements are being considered of such a size that there will be a large velocity change along the height.

P. S. Eagleson reported an interest in the effects of velocity gradients resulting from studies of sediment transport. A short study made of the drag on a sphere rolling down a plane showed a considerably different drag coefficient from that for flow without a velocity gradient. In the future it is hoped that an electro-magnetic balance can be constructed so that forces on a sphere suspended in a flow can be measured independent of the effects of mechanical friction.

Committee Objectives

It was decided that the committee should attempt

(i) To promote work in this subject in our own organizations and to demonstrate the need to national agencies and other organizations.

(ii) To support proposals made for funds for the carrying through of this work.

(iii) To arrange programs at ASCE meetings.

(iv) To collect a symposium of related papers as published in ASCE Proceedings.

The committee is interested in having its name changed. Inasmuch as many of the practical problems of flows with velocity gradients do not involve a boundary layer distribution e.g. wave forces, it was decided that the name of the committee should be changed. The name suggested was "Flow Around Objects in a Non-uniform Velocity Field."

Items for February Letter

If your editor receives sufficient material a news letter will be prepared for circulation with the EMD Journal dated February 1961. An attempt is being made to have the deadline for newsletter material set forward so as to increase its news value; however, until a decision is made to change, it will be necessary to operate under the current policy which requires that items for use in the February 1961 newsletter be on the editors desk by Thursday, December 29, 1960.

At the Annual Meeting of the EMD Executive Committee considerable interest was expressed in devoting more space here to editorial comments by members of the division. Your brief editorial comments as well as news items will enhance the value of the news letter and will be enthusiastically received.

Donald L. Dean
News Letter Editor EMD
Department of Civil Engineering
University of Delaware
Newark, Delaware

NEW DIRECTORY IS AVAILABLE TO MEMBERS

The 1960 Directory is now available to members on request. The Directory lists the entire membership of the Society, giving the membership grade,

position, and mailing address of each. In addition, there is a complete listing of the Honorary Members, past and present, and the Life Members. A useful geographical listing of the members is also included.

It goes without saying that the information contained in the Directory is of value to every member, and every member can obtain this valuable information. To receive your free copy of the Directory simply fill out the coupon below. Prompt delivery depends on prompt return of the coupon.

The Society publishes the membership Directory every other year. The next edition will be issued in 1962.

DIRECTORY 1960

ASCE members are entitled to receive, free of charge, the 1960 ASCE Directory. To obtain the directory simply clip this coupon and mail to: American Society of Civil Engineers, 33 West 39th Street, New York 18, N. Y.

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PAPERS FROM THE 2nd CONFERENCE ON ELECTRONIC COMPUTATION

The papers presented at the 2nd Conference on Electronic Computation in Pittsburgh, September 8-9, 1960, are being offered in a single hard-bound volume. This special edition is composed of the thirty two technical papers, the welcome address, keynote address and three luncheon addresses, all as delivered at the Conference. The price (post-paid) is as follows:

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